Tensor Algebra

1.- Cartesian Tensors

* Summation Convention: a sum whose summands are obtained giving values $1, 2, 3, \ldots, n$ to certain indices, we have indicated with the symbol Σ, together with the range of index variation.

For example:

$$\sum_{i=1}^{n} a_i = a_1 + a_2 + \cdots + a_n \tag{1}$$

$$\sum_{i=1}^{n} a_i b_i = a_1 b_1 + a_2 b_2 + \cdots + a_n b_n \tag{2}$$

$$\sum_{j=1}^{n} a_{ij} x_j = a_{i1} x_1 + a_{i2} x_2 + \cdots + a_{in} x_n \tag{3}$$

In tensor calculus appear very frequenly sums of the type of Eqs. (2) and (3), where we add on two repeated indices. In this case it is comfortable and the writing is abbreviated if we agree that for such indices the sum sign will be suppressed. The following is established:

Einstein convention: when in a monomia expression there are two repeated indices, it will be understood that it is a sum in which the repeated indices are added from 1 to n, where n is the dimension of space.

So, the examples (2) and (3) are simplified by the convention to:

$$a_i b_i = a_1 b_1 + a_2 b_2 + \cdots + a_n b_n$$

$$a_{ij} x_j = a_{i1} x_1 + a_{i2} x_2 + \cdots + a_{in} x_n$$

Remark 1: any free index in an expression shall have the same range as summation indices, unless stated otherwise.

Remark 2: No index may occur more than twice in any given expression.

Observation: the repeated indices can be represented with any letter without the sum changing. We have, for example:

$$a_i b_i = a_n b_n$$

because both members represent the sum

$$a_1 b_1 + a_2 b_2 + \cdots + a_n b_n$$

Example 1: (a) According to Remark 2, an expression like $a_{ii} x_i$ is without meaning.

(b) An expression of the form $a_i(x_i + y_i)$ is considered well-defined, for it is obtained by composition of the meaningful expression $a_i z_i$ and $x_i + y_i = z_i$. In other words, the index i is regarded as occuring once in the term $(x_i + y_i)$.

* Free and Dummy Indices

In the Eq. (3), the expression $a_{ij} x_j$ involves two sorts of indices:

* The index of summation, j, which ranges over the integers $1, 2, 3, \ldots, n$

  It is clear that the use of the particular character j is inessential, because the expressions $a_{ir} x_r$ and $a_{iv} x_v$ represent exactly the same sum as $a_{ij} x_j$ does.

  For this reason, j is called a dummy index.

* The index i, which may take on any particular value $1, 2, 3, \ldots, n$ independently.

  So, i is called a free index.

Example 2. If $n=3$, write down explicitly the equations represented by the expression $y_i = a_{ir} x_r$

Holding i fixed and summing over $r = 1, 2, 3$ yields

$$y_i = a_{i1} x_1 + a_{i2} x_2 + a_{i3} x_3$$

Next, setting the free index $i = 1, 2, 3$ leads to three separate equations

$$y_1 = a_{11} x_1 + a_{12} x_2 + a_{13} x_3$$
$$y_2 = a_{21} x_1 + a_{22} x_2 + a_{23} x_3$$
$$y_3 = a_{31} x_1 + a_{32} x_2 + a_{33} x_3$$

* Double Sums: an expression can involve more than one summation index For example, $a_{ij} x_i y_j$ indicates a summation taking place on both i and j silmutaneously. The expansion of $a_{ij} x_i y_j$ can be arrived by first summing over i, then over j:

$$a_{ij} x_i y_j = a_{ij} x_i y_j + a_{2j} x_2 y_j + a_{3j} x_3 y_j + \cdots + a_{nj} x_n y_j \quad \text{(summed over } i\text{)}$$
$$= (a_{11} x_1 y_1 + a_{12} x_1 y_2 + \cdots + a_{1n} x_1 y_n) + \quad \text{(summed over } j\text{)}$$
$$+ (a_{21} x_2 y_1 + a_{22} x_2 y_2 + \cdots + a_{2n} x_2 y_n) +$$
$$+ \cdots$$
$$+ (a_{n1} x_n y_1 + a_{n2} x_n y_2 + \cdots + a_{nn} x_n y_n)$$

The result is the same is one sums over j first, and then over i

Example 3. If $n=2$, write down explicitly the equations represented by the expression $y_i = c_i^r a_{rs} x_s$

First, summing over $r = 1, 2$.

$$y_i = c_i^1 a_{1s} x_s + c_i^2 a_{2s} x_s$$

And now summing over $s = 1, 2$

$$y_i = c_i^1 a_{11} x_1 + c_i^2 a_{21} x_1 + c_i^1 a_{12} x_2 + c_i^2 a_{22} x_2$$

Finally, setting the free index $i = 1, 2$.

$$y_1 = c_1^1 a_{11} x_1 + c_1^2 a_{21} x_1 + c_1^1 a_{12} x_2 + c_1^2 a_{22} x_2$$
$$y_2 = c_2^1 a_{11} x_1 + c_2^2 a_{21} x_1 + c_2^1 a_{12} x_2 + c_2^2 a_{22} x_2$$

* Substitutions :

Suppose it is required to substitute $y_i = a_{ij} x_j$ in the equation $Q = b_{ij} y_i x_j$:

$$Q = b_{ij} a_{ij} x_j x_j \quad \times$$

It is an absurd expression according to Remark 2

The correct procedure is:

Step 1: Identify any dummy indices in the expression to be substituted that coincide with indices occurring in the main expression

Main expression: $Q = b_{ij} y_i x_j$

Expression to be substituted: $y_i = a_{ij} x_j$

Dummy index j is duplicated

Step 2: Changing these dummy indices to characters not found in the main expression

$y_i = a_{ir} x_r$ Change dummy index from j to r

Step 3: Carry out the substitution in the usual fashion

$$Q = b_{ij} (a_{ir} x_r) x_j$$

$$= a_{ir} b_{ij} x_r x_j \qquad \text{Substitute and rearrange}$$

Example 4: if $y_i = a_{ij} x_j$, express the cuadratic form $Q = g_{ij} y_i y_j$ in terms of the x-variables.

First write:

$$Y_i = a_{ir} x_r$$
$$Y_j = a_{js} x_s$$

Then, by substitution,

$$Q = g_{ij} (a_{ir} x_r)(a_{js} x_s)$$
$$= g_{ij} a_{ir} a_{js} x_r x_s$$
$$= h_{rs} x_r x_s$$

where $h_{rs} \equiv g_{ij} a_{ir} a_{js}$

* Kronecker Delta and algebraic manipulations

A much used symbol in tensor calculus has the effect of annihilating the "off-diagonal" terms in a double summation

$$\delta_{ij} \equiv \delta^i_j \equiv \delta^{ij} \equiv \begin{cases} 1, & i = j \\ 0, & i \neq j \end{cases}$$

Clearly, $\delta_{ij} = \delta_{ji}$ for all i, j.

Example 5: If $n = 3$,

$$\delta_{ij} x_i x_j = \delta_{1j} x_1 x_j + \delta_{2j} x_2 x_j + \delta_{3j} x_3 x_j$$
$$= \delta_{11} x_1 x_1 + \delta_{22} x_2 x_2 + \delta_{33} x_3 x_3$$
$$= (x_1)^2 + (x_2)^2 + (x_3)^2$$
$$= x_i x_i$$

In general:

$$\delta_{ij} x_i x_j = x_i x_i$$
$$\delta^r_j a_{ir} x_i = a_{ij} x_i$$