$$H = -J \sum_{\langle ij \rangle} \sigma_i \sigma_j - h \sum_i \sigma_i$$

si $J = 0$ $\left(H = -h \sum_i \sigma_i \right)$

$\rightarrow$ fácil.

$$z = 2\,\mathrm{ch}(\beta h)$$

ahora $J \neq 0$

$\sigma_i = m + \delta\sigma_i$ donde $\langle \sigma_i \rangle = m$

$$\sigma_i \sigma_j = (m + \delta\sigma_i)(m + \delta\sigma_j)$$
$$= m^2 + m(\delta\sigma_i + \delta\sigma_j) + \cancel{\delta\sigma_i \delta\sigma_j}$$

$$\rightarrow H = \frac{N m^2 z J}{2} - h_{eff} \sum_i \sigma_i$$

$$h_{eff} = \underline{Jzm} + h$$

$$Z = \left[2\,ch(h_{eff}\beta)\right]^N e^{-\frac{Nm^2 zJ}{2}}$$

$$\underline{F = N\left(\frac{Jzm^2}{2} - k_B T \ln\left[2\,ch(\beta h_{eff})\right]\right)}$$

$$\left.\frac{F}{N}\right|_{h=0} \simeq -k_B T \ln 2 + \frac{Jzm^2}{2} - k_B T \ln\left[1 + \frac{(\beta zJm)^2}{2} + \dots\right]$$

$$h_{eff} = Jzm$$

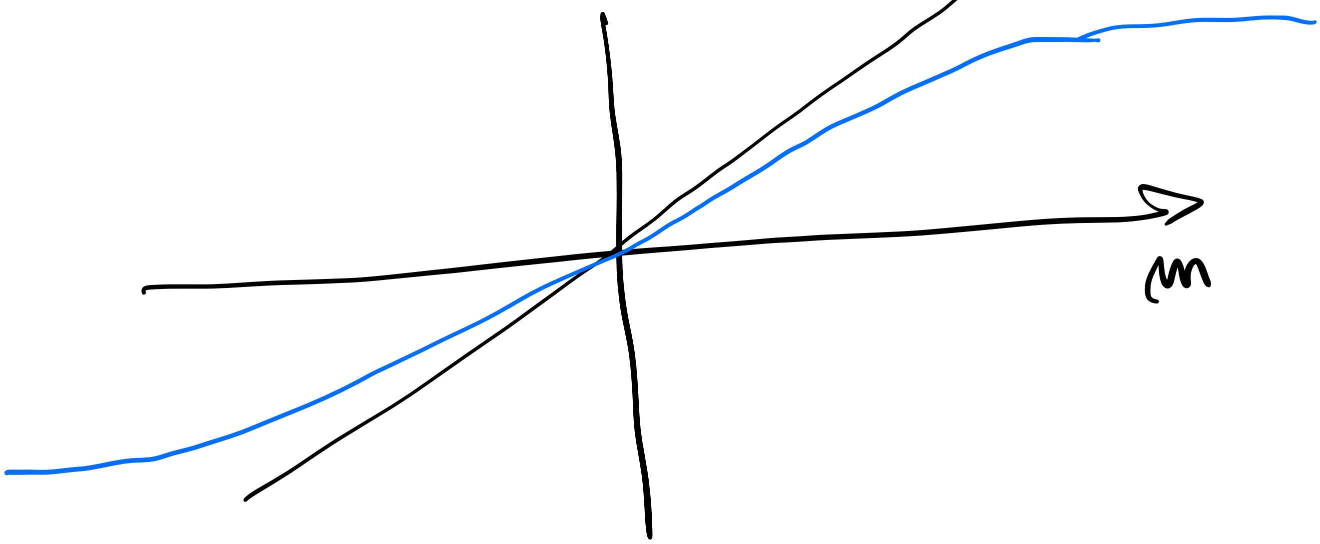
$$\boxed{\frac{F}{N} \simeq cte + \frac{Jz}{2}\underline{\left(1 - \frac{zJ}{k_B T}\right)}m^2 + \mathcal{O}(m^4)}$$

$$\boxed{\underset{\sim}{m} = \left.\langle \sigma_i \rangle\right|_{h=0} = \underline{tgh\left[\beta J z m\right]}}$$

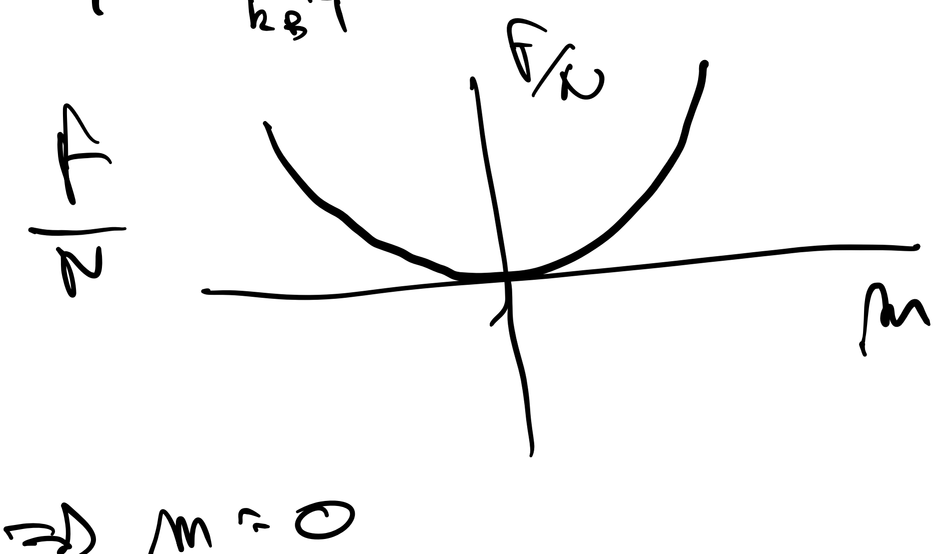
$T > T_c = \frac{k_B}{zJ}$

$T < T_c = \frac{k_B}{zJ}$

a) $T > T_c$

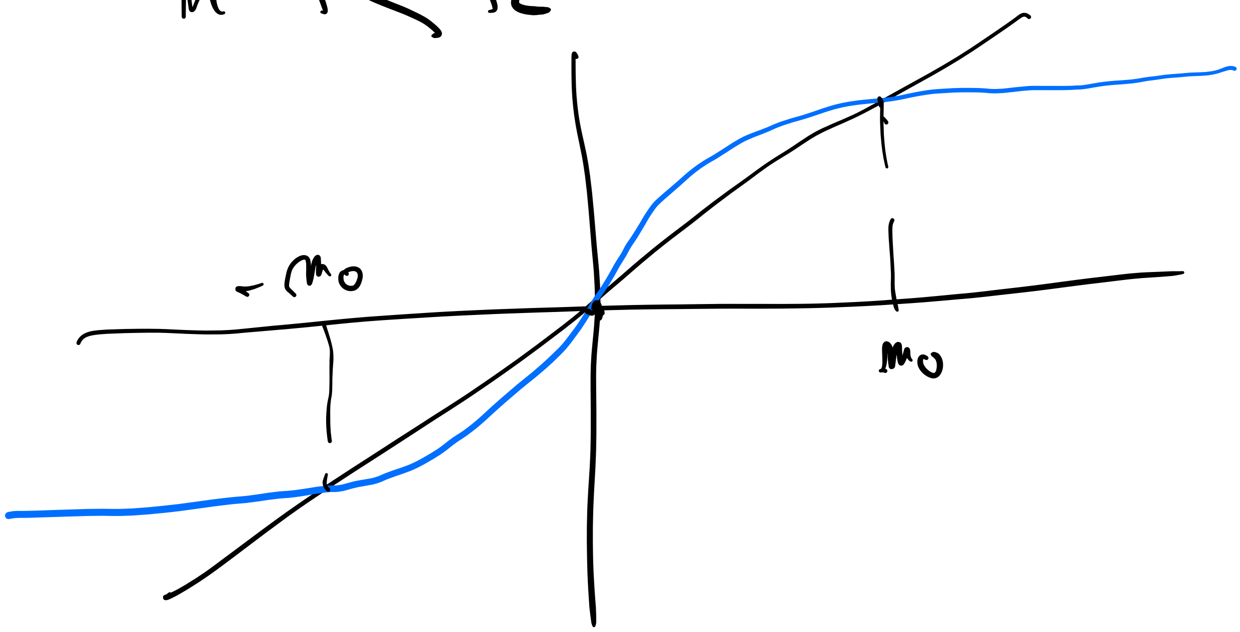


$$1 - \frac{zJ}{k_B T} > 0$$

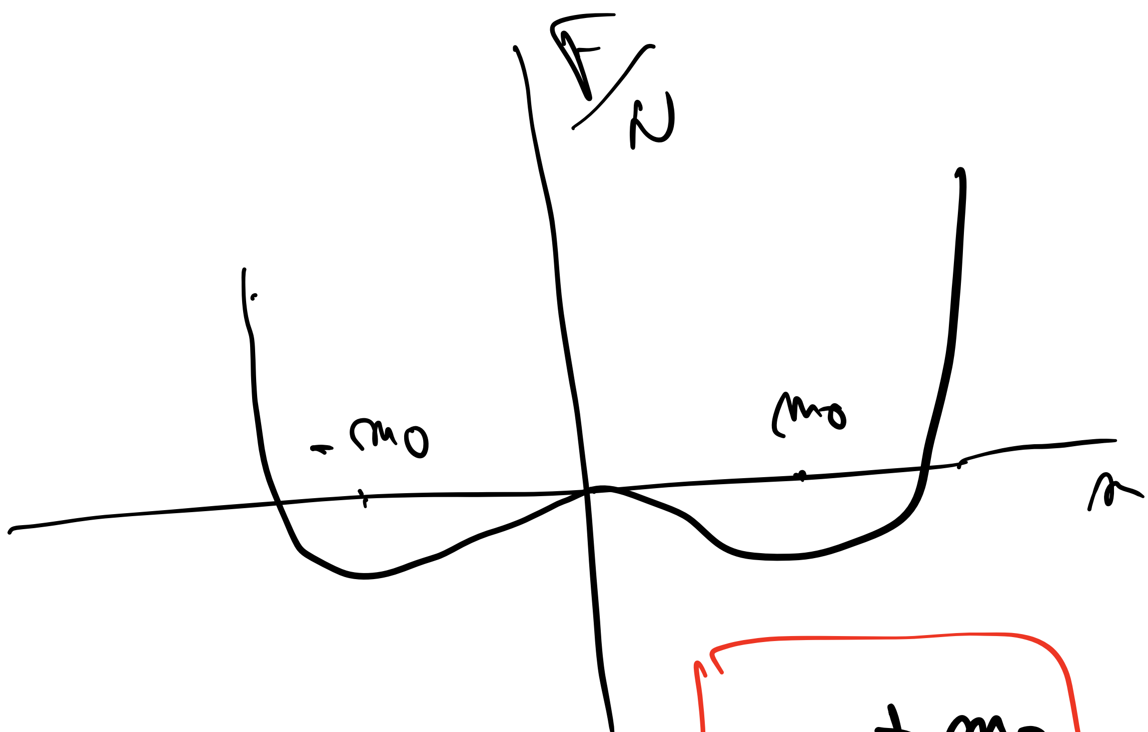


$\Rightarrow m = 0$

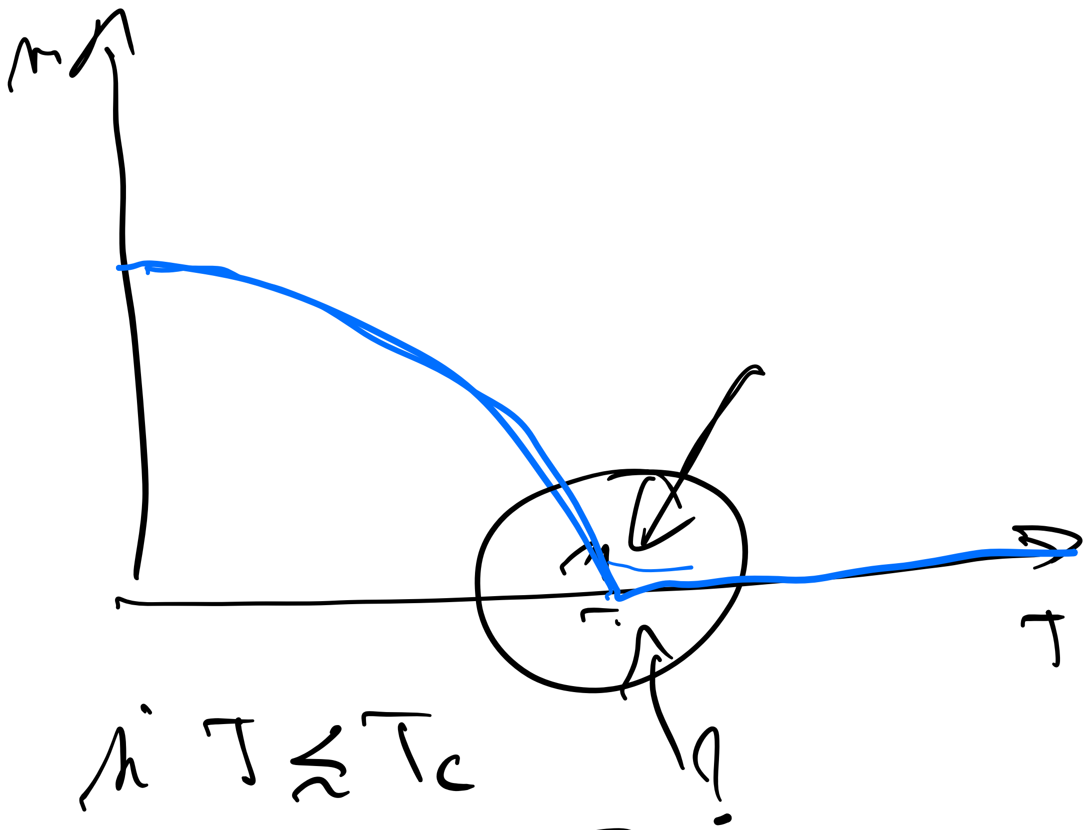
ii $T < T_c$



$$1 - \frac{zJ}{k_B T} < 0$$



$$\boxed{m = \pm m_0}$$



si $T \lesssim T_c$

$$m \sim \sqrt{(T_c - T)} \to \beta = \frac{1}{2}$$

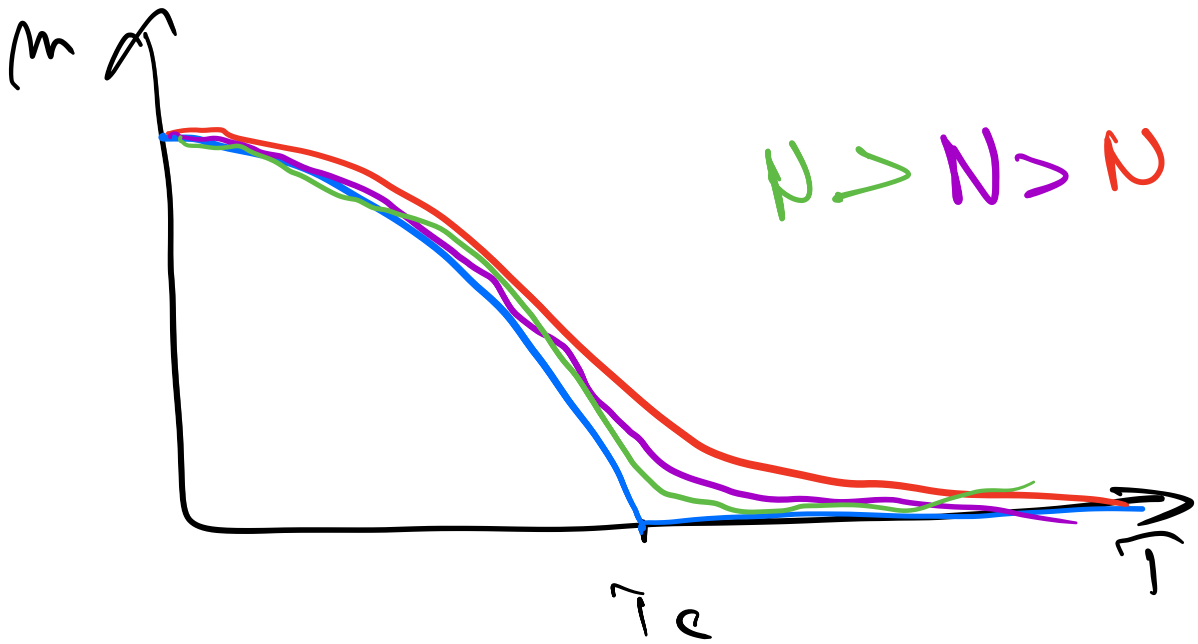
<u>obs</u>

$$Z = \sum e^{-\beta H}$$

Z no analítica si el # de config. $\longrightarrow \infty$

→ límite termodinámico.

$N \to \infty$



II Integral funcional para el modelo de Ising.

1) Transformación de Hubbard-Stratonovich

$$H = -\frac{1}{2}\sum_{i,j} \underline{J_{ij}}\, \underline{\sigma_i \sigma_j}$$

por ejemplo $J_{ij} \sim$ función de $|i-j|$
distancia

J_{ij} decae con $|i-j|$

$\Rightarrow$ escritura matricial

$$\rightarrow \sigma = \begin{pmatrix} \sigma_1 \\ \sigma_2 \\ \vdots \\ \sigma_N \end{pmatrix} \qquad \sigma^T = (\sigma_1, \sigma_2 \dots \sigma_N)$$

$$H = -\frac{1}{2}\sigma^T J \sigma$$

J matriz $N \times N$, real, simétrica

todos los autovalores de J son positivos

$\underline{J > 0}$!

$$\sigma_i = \pm 1 \Rightarrow \sigma_i^2 = 1$$

$$H = -\sum_{i,j} J_{ij}\, \sigma_i \sigma_j - C \sum_i \sigma_i^2$$

$$= -\sum_{\forall i,j} \tilde{J}_{ij} \sigma_i \sigma_j$$

donde $\tilde{J}_{ij} = J_{ij} + C\,\delta_{i,j}$

$$\tilde{J} = J + C\,\mathbb{1}$$

si λ es autovalor de J

$\lambda + c$ " " $\tilde{J}$

ej: Mostrar que si $A^t = A \in \mathbb{R}$ y $A > 0$

$$\underline{\underline{I}} = \int_{-\infty}^{\infty} \prod_{i=1}^{N} dx_i \, \underline{e^{-\frac{1}{2}\sum_{i,j} x_i A_{ij} x_j + \underline{\sum_{i=1}^{N} x_i y_i}}}$$

$y_1, y_2, \dots, y_N \in \mathbb{R}$

$$e^{-\frac{1}{2} x^T A x + x^T y}$$

$$I = \sqrt{\frac{(2\pi)^N}{\mathrm{Det}(A)}} \, e^{\frac{1}{2}\sum_{i,j} y_i (A^{-1})_{ij} y_j}$$

$$= \underbrace{\sqrt{\frac{(2\pi)^N}{\mathrm{Det}(A)}}}_{?} \, \underline{e^{\frac{1}{2} y^t A^{-1} y}}$$

Hint 1 :

$$X = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = \underline{\hat{x} + A^{-1} y}$$

$$x_i = \hat{x}_i + \sum_j (A^{-1})_{ij} y_j$$

$$\int \prod_i dx_i = \int \prod_i d\hat{x}_i$$

$$I = \underbrace{e^{\frac{1}{2} y^T A^{-1} y}} \times \underbrace{\int_{-\infty}^{\infty} \prod_{i=1}^{n} d\hat{x}_i \, e^{-\frac{1}{2} \hat{x}^T A \hat{x}}}_{\overset{?}{=}}$$

$$\exists\, O \,/\, \text{tq } \underline{O^T O = \mathbb{1}} = I_d = \begin{pmatrix} 1 & & 0 \\ & 1 & \\ 0 & & 1 \end{pmatrix}$$

$$O^T A O = D = \begin{pmatrix} \lambda_1 & & 0 \\ & \lambda_2 & \\ 0 & & \ddots \; \lambda_n \end{pmatrix}$$

Cambio de variables

$$\hat{x} = \theta \hat{\tilde{x}} \quad (\text{Jacobiana} = 1)$$

$$I = e^{\frac{1}{2} y^T A^{-1} y} \int_{-\infty}^{\infty} \prod_i d\hat{\tilde{x}}_i \, e^{-\sum_{i=1}^{N} d_i \hat{\tilde{x}}_i^2}$$

$$\int_{-\infty}^{\infty} e^{-\alpha x^2} dx = \sqrt{\frac{2\pi}{\alpha}}$$

$$\boxed{\int_{-\infty}^{\infty} \prod_i dx_i \, e^{-\frac{1}{2} x^T A x + x^T y} = \sqrt{\frac{(2\pi)^N}{\mathrm{Det}(A)}} \, e^{\frac{1}{2} y^T A^{-1} y}}$$

Modelo de Ising

$$H = -\frac{1}{2} \sigma^T J \sigma \qquad \frac{\beta}{2} J \Longleftrightarrow A^{-1}$$

$$Z = \sum_{\{\sigma\}} e^{\frac{\beta}{2}\sigma^T J \sigma}$$

$$e^{\frac{\beta}{2}\sigma^T J \sigma} = (2\pi)^{-N/2}\left[\mathrm{Det}(\beta J)\right]^{-1/2} \times \int_{-\infty}^{\infty} \prod_{i=1}^{N} d\phi_i \, e^{-\frac{1}{2\beta}\phi^T J^{-1}\phi + \underline{\sigma^T\phi}}$$

$$\Rightarrow Z = (2\pi)^{-\frac{N}{2}}\left[\mathrm{Det}(\beta J)\right]^{-1/2} \times \int_{-\infty}^{\infty} \prod_{i} d\phi_i \, e^{-\frac{1}{2\beta}\phi^T J^{-1}\phi} \underline{\sum_{\{\sigma_i = \pm 1 \ \forall i\}} e^{\sum_{i=1}^{N}\sigma_i\phi_i}}$$

obs

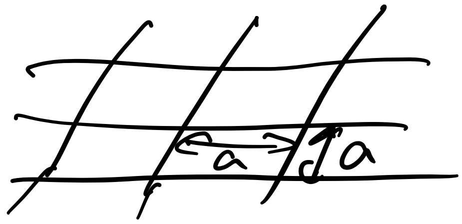
$$\sum_{\{\sigma\}} e^{\sum_i \sigma_i \phi_i} = \prod_i \left(\sum_{\sigma_i = \pm 1} e^{\sigma_i \phi_i} \right)$$

$$= \prod_i 2\,\mathrm{ch}(\phi_i)$$

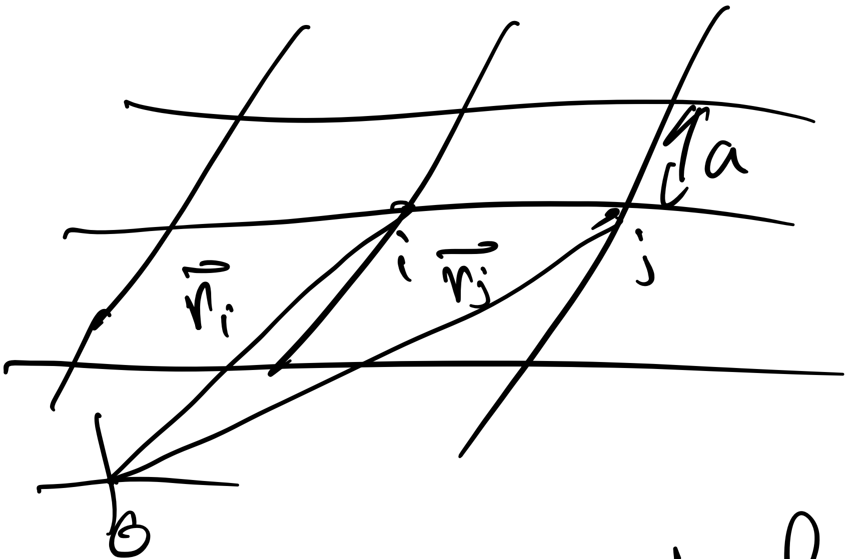
$$\boxed{\Rightarrow Z = (2\pi)^{-N/2} \left[\mathrm{Det}(\beta J) \right]^{-1/2} \times \int_{-\infty}^{\infty} \prod_i d\phi_i \; e^{-\frac{1}{2\beta} \phi^T J^{-1} \phi + \sum_i \ln(2\,\mathrm{ch}\,\phi_i)}}$$

2) Limite al continuo.

A $T \gtrsim T_c$, $\xi \to \infty$



el régimen $\xi \gg a$
$\Rightarrow$ límite del continuo



$$\sum_i \longrightarrow \frac{1}{a^d}\int d^d \vec{r}_i$$

$$\delta_{i,j} \longrightarrow a^d\, \delta^d(\vec{r}_i - \vec{r}_j)$$

$$\sum_i \delta_{i,j} = 1 \longrightarrow \frac{1}{a^d}\int d^d \vec{r}_i\, a^d\, \delta^d(\vec{r}_i - \vec{r}_j) = 1$$

$$J_{i,j}\,,\quad J^{-1}_{i,j}$$

$$J_{i,j} \longrightarrow J(\vec{r}_i, \vec{r}_j) \rightarrow J(|\vec{r}_i - \vec{r}_j|)$$

$$J^{-1}_{i,j} \longrightarrow J^{-1}(|\vec{r}_i - \vec{r}_j|)$$

inv. por traslación

$$\sum_l J_{il} J^{-1}_{lj} = \delta_{i,j}$$

$$\hookrightarrow \frac{1}{a^d}\int d^d\vec{r}\; J(|\vec{r} - \vec{r}_i|)\, J^{-1}(|\vec{r} - \vec{r}_j|)$$

$$= a^d\, \delta(\vec{r}_i - \vec{r}_j) \qquad \boxed{(*)}$$

$$\phi_i \rightarrow \phi(\vec{r})$$

$$\sum_i \phi_i \longrightarrow \frac{1}{a^d}\int d^d\vec{r}\, \phi(\vec{r})$$

$$\hat{\phi}(\vec{k}) = \int d^d\vec{r}\, e^{i\vec{k}\cdot\vec{r}}\, \phi(\vec{r})$$

$$\phi(\vec{r}) = \frac{1}{(2\pi)^d}\int d^d\vec{k}\, e^{-i\vec{k}\cdot\vec{r}}\, \hat{\phi}(\vec{k})$$

obs $\phi(\vec{r}) \in \mathbb{R} \Rightarrow \hat{\phi}(-\vec{k}) = \hat{\phi}^*(\vec{k})$

$$\hat{J}(\vec{k}) = \int d^d\vec{r}\, e^{i\vec{k}\cdot\vec{r}}\, J(\vec{r})$$

$$J(\vec{r}) = \frac{1}{(2\pi)^d}\int d\vec{k}\, e^{-i\vec{k}\cdot\vec{r}}\, \hat{J}(\vec{k})$$

$$\hat{J}^{-1}(\vec{k}) = \int d^d\vec{r}\, e^{i\vec{k}\cdot\vec{r}}\, J^{-1}(\vec{r})$$

$$\int d\vec{r}\, e^{i(\vec{k}-\vec{k}')\cdot\vec{r}} = (2\pi)^d\, \delta(\vec{k}-\vec{k}')$$

$$\int d\vec{k}\, e^{i(\vec{r}-\vec{r}')\cdot\vec{k}} = (2\pi)^d\, \delta(\vec{r}-\vec{r}')$$

$$(*) \quad \boxed{\hat{J}^{-1}(\vec{k}) = \frac{a^{2d}}{\hat{J}(\vec{k})}}$$

$$\sum_{i,j} \phi_i J^{-1}_{ij} \phi_j$$

$$\rightarrow \frac{1}{a^{2d}} \int dr\, d\vec{r}\,' \, \phi(\vec{r})\, J^{-1}(\vec{r}-\vec{r}\,')\, \phi(\vec{r}\,')$$

$$= \frac{1}{a^{2d}} \frac{1}{(2\pi)^d} \int d\vec{k}\, |\hat{\phi}(\vec{k})|^2\, \hat{J}^{-1}(\vec{k})$$

$$= \frac{1}{(2\pi)^d} \int d\vec{k}\, \frac{|\hat{\phi}(\vec{k})|^2}{\hat{J}(\vec{k})}$$

$$\sum_i \ln[\mathrm{ch}(\phi_i)] \rightarrow \frac{1}{a^d} \int d\vec{r}\, \ln[\mathrm{ch}(\phi(\vec{r}))]$$

$$\hat{S}(\vec{k}) = \hat{S}(\vec{0}) + \cancelto{0}{\sum_{\alpha} k^{\alpha} \frac{\partial \hat{S}}{\partial k^{\alpha}}\Big|_{\vec{k}=0}}$$

$$+ \frac{1}{2} \sum_{\alpha,\beta} k^{\alpha} k^{\beta} \frac{\partial^2 \hat{S}}{\partial k^{\alpha} \partial k^{\beta}}\Big|_{\vec{k}=0} + \dots$$

Obs $\frac{\partial \hat{S}}{\partial k^{\alpha}}\Big|_{\vec{k}=0}$

$$\hat{S}(\vec{k}) = \int d\vec{r}\, e^{i\vec{k}\cdot\vec{r}}\, S(|\vec{r}|)$$

$$\frac{\partial \hat{S}}{\partial k^{\alpha}}\Big|_{\vec{k}=0} = \int d\vec{r}\; i r^{\alpha} \underbrace{S(|\vec{r}|)}_{r^{\alpha} \to -r^{\alpha}} = 0$$

$$\frac{\partial^2 \hat{S}}{\partial k^{\alpha} \partial k^{\beta}}\Big|_{\vec{k}=0} = -\underbrace{\int d\vec{r}\; r^{\alpha} r^{\beta} S(|\vec{r}|)}_{=0 \text{ si } \alpha \neq \beta}$$

$$\hat{J}(\vec{k}) = \hat{J}(0) - \frac{k^2}{2}\underline{\int \frac{\vec{r}^2}{d} J(\vec{r})} + \dots$$

def ℓ = alcance de J

$$\ell^2 = \frac{\frac{1}{2d}\int d\vec{r}\, \vec{r}^2 J(\vec{r})}{\int d\vec{r}\, J(\vec{r})}$$

$$\Rightarrow \hat{J}(\vec{k}) = \hat{J}(0)\underline{\left[1 - \ell^2 k^2 + \dots\right]}$$

$$\frac{1}{\hat{J}(\vec{k})} = \frac{1}{\hat{J}(0)}\left(1 + \ell^2 k^2 + \dots\right)$$

$$\frac{1}{2\beta}\phi^T J^{-1}\phi = \frac{1}{(2\pi)^d}\frac{1}{2\beta}\int d\vec{k}\, \frac{|\hat{\phi}(\vec{k})|^2}{\hat{J}(\vec{k})}$$

$$\simeq \frac{1}{2\beta\hat{J}(0)}\int \frac{d\vec{k}}{(2\pi)^d}\, |\hat{\phi}(\vec{k})|^2\left(1 + \ell^2 k^2 + \dots\right)$$

$$= \frac{1}{2\beta\,\hat{\xi}(0)} \int d\vec{r} \left[\phi^2(\vec{r}) + \ell^2 |\vec{\nabla}\phi|^2 + \dots \right]$$

$$\ln\left[ch\,\phi(\vec{r})\right] \simeq \ln\left(1 + \frac{\phi^2}{2} + \frac{\phi^4}{4!} + \dots\right)$$

$$\simeq \frac{\phi^2}{2} - \frac{\phi^4}{12} + \dots$$

$$e^{-\frac{1}{2\beta}\phi^T \xi^{-1}\phi + \sum_i \ln[2ch\phi_i]}$$

$$\longrightarrow 2^N e^{-\frac{1}{2\beta\hat{\xi}(0)}\int d\vec{r}\left[\phi^2(\vec{r}) + \ell^2|\vec{\nabla}\phi|^2 + \dots\right]}$$

$$\times\, e^{+\frac{1}{a^d}\int d\vec{r}\left[\frac{\phi^2}{2}(\vec{r}) - \frac{\phi^4}{12}(\vec{r})\right]}$$

$$\Rightarrow Z = cte \int \mathcal{D}\{\phi\}\, e^{-S\{\phi\}}$$

donde $cte = (2\pi)^{-N/2}\left[Det(\beta\xi_{ij})\right]^{-1/2} 2^N$

$$\mathcal{D}\{\phi\} = \lim_{N\to\infty} \prod_i d\phi_i$$

$$S\{\phi\} = \int d\vec{r}\left[\frac{c}{2}|\vec{\nabla}\phi|^2 + \frac{a_2}{2}\phi^2(\vec{r}) + a_4\phi^4(\vec{r}) + \dots\right]$$

$$c = \frac{\ell^2}{\beta\hat{\xi}(0)}\ ;\quad \boxed{a_2 = \frac{k_B T}{\hat{\xi}(0)} - \frac{1}{a^d}}\ ;\quad a_4 = \frac{1}{12a^d}$$

acción de Ginzburg-Landau

$$S[\phi] = \int d\vec{n} \left[\frac{c}{2} |\vec{\nabla}\phi|^2 + \frac{a_2}{2}\phi^2 + a_4\phi^4 + \dots \right]$$

$$S[\phi] = \int d^4x \left[\eta^{\mu\nu}\partial_\mu\phi\,\partial_\nu\phi + m^2\phi^2 + \lambda\phi^4 \right]$$

$$\eta^{\mu\nu} = \begin{pmatrix} -1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}$$

<u>abs</u>

$$\frac{F_{em}}{N} \sim cte\, \frac{Jz}{2}\left(1 - \frac{zJ}{k_B T}\right) m^2 + c\, m^4 + \dots$$

<u>abs</u> para minimizar $S[\phi]$

$$\vec{\nabla}\phi = \vec{0} \Rightarrow \phi = m$$

$$S[m] \simeq V \left[\frac{a_2}{2} m^2 + a_4 m^4 + \dots \right]$$