La Conga I

Thursday, 14 July 2022 08:12

Cosmología del Big Bang

* Bases observacionales y teóricas
* Expansión del Universo
* Introducción al algebra tensorial
* Introducción a la Relatividad General

Bases Observacionales:

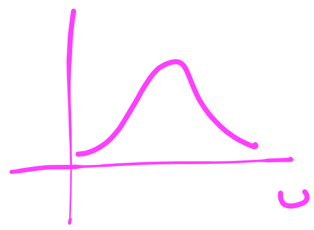
* Expansión del Universo

$$V = H\,d$$

$$H_0 = 74 \ km/s\,Mpc$$

* La existencia de la radiación cósmica de fondo

$T = 2.73\ K$

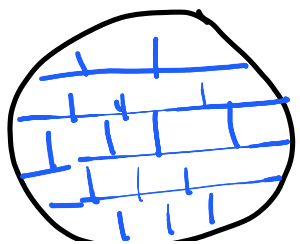


* La nucleosíntesis primordial

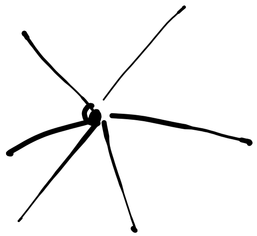
Bases teóricas.

La métrica de FRW:

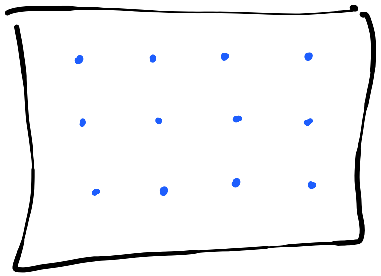
Homogeneidad



Isotropía:



Homogéneo e isótropo

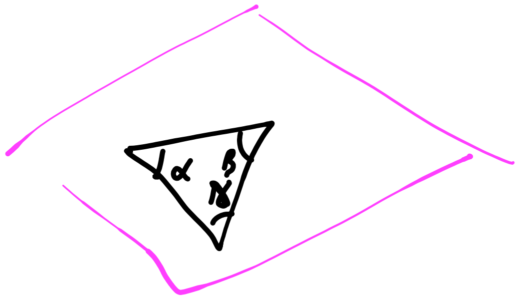


FRW:

$$ds^2 = -dt^2 + a^2(t)\left[\frac{dr^2}{1-kr^2} + r^2\left(d\theta^2 + \sin^2\theta\, d\phi^2\right)\right]$$

$$k = \begin{cases} -1 \\ +1 \\ 0 \ \checkmark \end{cases}$$

A flat : $K = 0$



$$\alpha + \beta + \gamma = \pi$$

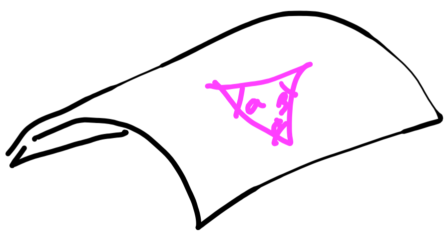
$$ds^2 = dx^2 + dy^2$$

Superficie de una esfera.

$$\alpha+\beta+\gamma > \pi$$

$$ds^2 = dr^2 + R^2 \sin^2(r/R)\, d\sigma^2$$

El hyperboloide:



$$\alpha + \beta + \gamma < \pi$$

$$ds^2 = dr^2 + R^2 \sinh^2(r/R)\, d\theta^2$$

* Álgebra tensorial

* Convensión de suma

$$\sum_{n} \quad , 1, 2, 3, \ldots, n$$

$$\sum_{i=1} a_i = a_1 + a_2 + \cdots + a_n$$

$$\sum_{i=1}^{n} a_i b_i = a_1 b_1 + a_2 b_2 + \cdots + a_n b_n$$

$$\sum_{j=1}^{n} a_{ij} x_j = a_{i1} x_1 + a_{i2} x_2 + \cdots + a_{in} x_n$$

* Convesión de Einstein.

$$a_i b_i = a_1 b_1 + a_2 b_2 + \cdots + a_n b_n$$

$$a_{ij} x_j = a_{i1} x_1 + a_{i2} x_2 + \cdots + a_{in} x_n$$

$$a_{ij} x_j = a_{il} x_l$$

i = libre
j = mudo

Ejemplo: Si $n=3$, escriba explícitamente la ecuación que está representada por la expresión $y_i = a_{ir} x_r$

$$y_i = a_{i1} x_1 + a_{i2} x_2 + a_{i3} x_3$$

$$y_1 = a_{11} x_1 + a_{12} x_2 + a_{13} x_3$$
$$y_2 = a_{21} x_1 + a_{22} x_2 + a_{23} x_3$$
$$y_3 = a_{31} x_1 + a_{32} x_2 + a_{33} x_3$$

Doble suma:

$$a_{ij}x_iy_j = a_{i1}x_iy_1 + a_{i2}x_iy_2 + \dots + a_{in}x_iy_n$$

$$= a_{11}x_1y_1 + a_{21}x_2y_1 + \dots a_{n1}x_ny_1$$
$$+ a_{12}x_1y_2 + a_{22}x_2y_2 + \dots a_{n2}x_ny_2$$
$$+ a_{1n}x_1y_n + \cdot a_{2n}x_2y_n + \cdot \quad a_{nn}x_ny_n$$

Ejemplo: si $n=2$, escriba explícitamente las ecuaciones representan la expresión $y_i = c_i^r a_{rs} x_s$

$r = 1,2$
$s = 1,2$

r: $y_i = c_i^1 a_{1s} x_s + c_i^2 a_{2s} x_s$

s: $y_i = c_i^1 a_{11} x_1 + c_i^2 a_{12} x_2 + c_i^2 a_{21} x_1 + c_i^2 a_{22} x_2$

$$Y_1 = C_1^1 a_{11} x_1 + C_1^2 a_{12} x_2 + C_1^2 a_{21} x_1 + C_1^2 a_{22} x_2$$

$$Y_2 = C_2^1 a_{11} x_1 + C_2^2 a_{12} x_2 + C_2^2 a_{21} x_1 + C_2^2 a_{22} x_2$$

Delta de Kronecker:

$$\delta_{ij} \equiv \delta_{ji} = \delta^{ij} = \begin{cases} 1, & i = j \\ 0, & i \neq j \end{cases}$$

$\delta_{ij} = \delta_{ji}$ para todo i, j

Ejemplo: Si $n = 3$

$$\delta_{ij} x_i x_j = \delta_{11} x_1 x_1 + \delta_{22} x_2 x_2 + \delta_{33} x_3 x_3$$

$$= (x_1)^2 + (x_2)^2 + (x_3)^2$$

$$= x_i x_i$$

$$\delta_{ij} x_i x_j = x_i x_i$$

$$\delta_j^r a_{ir} x_i = a_{ij} x_i$$

* Introducción a la Relatividad General:

La métrica

Es un tensor simétrico covariante de rango 2

$g_{\mu\nu}(x)$ define una métrica

$$ds'^2 = g_{\mu\nu}(x)\, dx^\mu dx^\nu$$ elemento de línea

* $g_{\mu\nu} = g_{\nu\mu}$

* $g = \det(g_{\mu\nu})$

* $g_{\mu\nu} g^{\nu\sigma} = \delta^\sigma_\mu$

* $T_\mu = g_{\mu\nu} T^\nu$

  $T^\mu = g^{\mu\nu} T_\nu$

Theorem: si la matrix Jacobiana de la transformación de un sistema de coordenadas dado (x^i) a un sistema rectangular (x'^i) is $J = \frac{\partial x'^i}{\partial x^j}$, entonces la matrix vectores ...

$g \equiv (g_{ij})$ del tensor métrico Euclídeo en el sistema (x^i)

está dado por:

$$g = J^T J$$

Ejemplos:

* El tensor métrico: la fórmula para la métrica de Minkowski está dada por

$$\eta_{\alpha\beta} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

* Relatividad especial

Signatura: $- + + + = +2$

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu$$

$$= g_{0\nu} dx^0 dx^\nu + g_{1\nu} dx^1 dx^\nu + g_{2\nu} dx^2 dx^\nu + g_{3\nu} dx^3 dx^\nu$$

$$= g_{00} dx^0 dx^0 + g_{11} dx^1 dx^1 + g_{22} dx^2 dx^2 + g_{33} dx^3 dx^3$$

$$= -(dx^0)^0 + (dx^1)^2 + (dx^2)^2 + (dx^3)^2$$

$x^0 = t$

$x^1 = x$

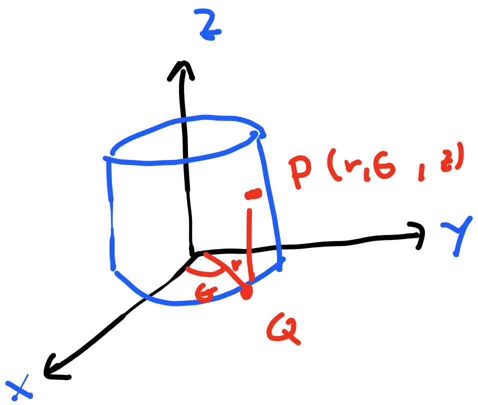
$x^2 = y$

$x^3 = z$

$$ds^2 = -dt^2 + dx^2 + dy^2 + (dz^2)$$

Como medir distancias!

* Calcular la métrica en coordenadas cilíndricas:



$$x = r\cos\theta$$
$$y = r\sin\theta$$
$$z = z$$

$$J = \begin{pmatrix} \cos\theta & -r\sin\theta & 0 \\ \sin\theta & r\cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$g = J^T J$$

$$\begin{pmatrix} \cos\theta & \sin\theta & 0 \end{pmatrix} \begin{pmatrix} \cos\theta & -r\sin\theta & 0 \end{pmatrix}$$

$$= \begin{pmatrix} -r\sin\theta & r\cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \sin\theta & r\cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$ds^2 = dr^2 + r^2 d\theta^2 + dz^2$$

* Los símbolos de Christoffel

$$\Gamma^{\alpha}_{\mu\nu} = \frac{1}{2} g^{\alpha\beta} \left(g_{\beta\mu,\nu} + g_{\beta\nu,\mu} - g_{\mu\nu,\beta} \right)$$

$$\Gamma^{\alpha}_{\mu\nu} = \Gamma^{\alpha}_{\nu\mu}$$

* Tensor de Curvatura: (Riemman tensor)

$$R^{\alpha}_{\beta\mu\nu} = \Gamma^{\alpha}_{\beta\nu,\mu} - \Gamma^{\alpha}_{\beta\mu,\nu} + \Gamma^{\alpha}_{\sigma\mu}\Gamma^{\sigma}_{\beta\nu} - \Gamma^{\alpha}_{\sigma\nu}\Gamma^{\sigma}_{\beta\mu}$$

$$R^{\alpha}_{\beta\mu\nu} = - R^{\alpha}_{\beta\nu\mu}$$

$R^{\alpha}_{\beta\mu\nu} = 0 \iff$ El espacio tiempo es plano

* Tensor de Ricci y Escalar de Ricci

Tensor de Ricci: $R_{\alpha\beta} = R^{\mu}{}_{\alpha\mu\beta} = R_{\beta\alpha}$

$$R_{\alpha\beta} = \Gamma^{\mu}_{\alpha\beta,\mu} - \Gamma^{\mu}_{\alpha\mu,\beta} + \Gamma^{\mu}_{\sigma\mu}\Gamma^{\sigma}_{\alpha\beta} - \Gamma^{\mu}_{\sigma\beta}\Gamma^{\sigma}_{\alpha\mu}$$

Escalar de Ricci:

$$R = g^{\mu\nu} R_{\mu\nu}$$

El tensor de Einstein.

$$G^{\mu\nu} = R^{\mu\nu} - \frac{1}{2} g^{\mu\nu} R = G^{\nu\mu}$$

Ejemplo: tensor de Riemann para coordenadas esféricas

$$R^{\alpha}{}_{\beta\mu\nu} = \Gamma^{\alpha}{}_{\beta\nu,\mu} - \Gamma^{\alpha}{}_{\beta\mu,\nu} + \Gamma^{\alpha}{}_{\sigma\mu}\Gamma^{\sigma}{}_{\beta\nu} - \Gamma^{\alpha}{}_{\sigma\nu}\Gamma^{\sigma}{}_{\beta -}$$

6 componentes que no son identicamente igual a zero

$$g_{\mu\nu} = \begin{pmatrix} 1 & & \\ & r^2 & \\ & & r^2 \sin^2\theta \end{pmatrix}$$

$$g^{\mu\nu} = \begin{pmatrix} 1 & & \\ & 1/r^2 & \\ & & 1/r^2 \sin^2\theta \end{pmatrix}$$

$$\Gamma^1_{22}\,,\quad \Gamma^1_{33}\,,\quad \Gamma^2_{12} = \Gamma^2_{21}\,,\quad \Gamma^2_{33}$$

$$\Gamma^3_{13} = \Gamma^3_{31}\,,\quad \Gamma^3_{23} = \Gamma^3_{32}$$

$$\Gamma^1_{22} = \frac{1}{2} g^{10}\left(g_{02,2} + \cancel{g_{12,2}}^{\,0} - g_{22,0}\right)$$

$$= \frac{1}{2} g^{11}\left(\cancel{g_{12,2}}^{\,0} - g_{22,1}\right)$$

$$= -\frac{1}{2} g^{11}\, g_{22,1}$$

$$= -\frac{1}{2} \frac{\partial}{\partial r}\left(r^2\right)$$

$$= -\frac{1}{2}(2r)$$

$$= -r$$

$$\Gamma^1_{33} = -r\sin^2\theta$$

$$\Gamma^2_{12} = \Gamma^2_{21} = \frac{1}{r}$$

$$\Gamma^2_{33} = -\sin\theta\cos\theta$$

$$\Gamma^3_{13} = \Gamma^3_{31} = \frac{1}{r}$$

$$\Gamma^3_{23} = \Gamma^3_{32} = \cot\theta$$

$$R^{1}{}_{212} = \Gamma^{1}{}_{22,1} - \cancel{\Gamma^{1}{}_{21,2}} + \underline{\Gamma^{1}{}_{\sigma 1}\Gamma^{\sigma}{}_{22}} - \underline{\Gamma^{1}{}_{\sigma 2}\Gamma^{\sigma}{}_{21}}$$

$$= \Gamma^{1}{}_{22,1} - \cancel{\Gamma^{1}{}_{21,2}} + \Gamma^{1}{}_{11}\cancel{\Gamma^{1}{}_{22}} + \Gamma^{1}{}_{21}\cancel{\Gamma^{2}{}_{22}}$$

$$+ \Gamma^{1}{}_{31}\cancel{\Gamma^{3}{}_{21}} - \cancel{\Gamma^{1}{}_{12}}\Gamma^{1}{}_{21} - \underline{\Gamma^{1}{}_{22}}\Gamma^{2}{}_{21} - \cancel{\Gamma^{1}{}_{32}}\Gamma^{3}{}_{21}$$

$$= \frac{d}{dr}(-r) + r\left(\frac{1}{r}\right)$$

$$= -1 + 1$$

$$= 0$$

$$R^{\alpha}{}_{\beta\mu\nu} = 0$$

* Ecuaciones de campo de Einstein

$$G^{\mu\nu} = 8\pi G\, T^{\mu\nu}$$

$$\begin{tikzcd} T^{\mu\nu} \arrow[r, bend left] & G_{\mu\nu} \arrow[l, bend left] \end{tikzcd}$$

$G = c = 1$

$$G^{\mu\nu} = 8\pi\, T^{\mu\nu}$$

$$1 = \frac{G}{c^2} = 7.425 \times 10^{-28} \text{ m/kg}$$

Constante	SI valor	Valor geometrizadas
c	2.998×10^{8} m/s	1
G	6.674×10^{-11} $m^3/kg\,s^2$	1
m_e	9.109×10^{-31} kg	6.764×10^{-58} m
m_l	1.673×10^{-27} kg	$1{,}673 \times 10^{-54}$ m