La Conga II

Friday, 22 July 2022 08:03

* Resolver las ecuaciones de campo de Einstein para la métrica de FRW

Métrica de FRW:

$$ds^2 = -dt^2 + a^2(t)\left[\frac{dr^2}{1-kr^2} + r^2\, d\theta^2 + r^2 \sin^2\theta\, d\phi^2\right]$$

Ecuación de Friedmann:

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi}{3}\rho - \frac{k}{a^2}$$

Ecuación de la aceleración

$$\frac{\ddot{a}}{a} = -\frac{4\pi}{3}(\rho + 3p)$$

Ecuaciones de campo de Einstein.

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R$$

$$R_{\beta\nu} = \Gamma^{\mu}_{\beta\nu,\mu} - \Gamma^{\mu}_{\beta\mu,\nu} + \Gamma^{\mu}_{\sigma\mu} \Gamma^{\sigma}_{\beta\nu} - \Gamma^{\mu}_{\sigma\nu} \Gamma^{\sigma}_{\beta\mu}$$

Símbolos de Christoffel:

$$\Gamma^{\alpha}_{\mu\nu} = \frac{1}{2} g^{\alpha\beta}(g_{\beta\mu,\nu} + g_{\beta\nu,\mu} - g_{\mu\nu,\beta})$$

$$g_{\alpha\beta} = \begin{pmatrix} -1 & & & \\ & \frac{a^2(t)}{1-kr^2} & & \\ & & a^2(t)\, r^2 & \\ & & & a^2(t)\, r^2 \sin^2\theta \end{pmatrix}$$

$$\Gamma^0_{\ 00} = \frac{1}{2} g^{0\beta}\left(g_{\beta 0,0} + g_{\beta 0,0} - g_{00,\beta}\right)$$

$$= \frac{1}{2} g^{00}\left(g_{\cancel{00},\cancel{0}} + g_{\cancel{00},0} - g_{\cancel{00},0}\right)$$

$$= 0$$

$$\Gamma^0_{\ 11} = \frac{1}{2} g^{0\beta}\left(g_{\beta 1,1} + g_{\beta 1,1} - g_{11,\beta}\right)$$

$$= \frac{1}{2} g^{00}\left(g_{0\cancel{1},1} + g_{\cancel{0}1,1} - g_{11,0}\right)$$

$$= \frac{1}{2}(-1)\left[-\frac{\partial}{\ }\left(\frac{a^2}{\ }\right)\right]$$

$$= \frac{1}{2}(-1)\left[-\frac{\partial}{\partial t}\left(\frac{\quad}{1-kr^2}\right)\right]$$

$$= \frac{1}{2}\,\frac{2a\dot{a}}{1-kr^2}$$

$$\Gamma^0_{11} = \frac{a\dot{a}}{1-kr^2}$$

$$\Gamma^0_{22} = \frac{1}{2}\, g^{0\beta}\left(g_{\beta 2,2} + g_{\beta 2,2} - g_{22,\beta}\right)$$

$$= \frac{1}{2}\, g^{00}\left(\cancel{g_{02,2}} + \cancel{g_{02,2}} - g_{22,0}\right)$$

$$= \frac{1}{2}(-1)\left[-\frac{\partial}{\partial t}\left(a^2(t)\, r^2\right)\right]$$

$$= a\dot{a}r^2$$

$$\Gamma^0_{33} = \frac{1}{2} g^{0\beta} \left(g_{\cancel{\beta}3,3} + g_{\beta \cancel{3},3} - g_{33,\beta} \right)$$

$$= \frac{1}{2} g^{00} \left(- g_{33,0} \right)$$

$$= -\frac{1}{2} \left\{ - \frac{\partial}{\partial t} \left[a^2(t)\, r^2 \sin^2\theta \right] \right\}$$

$$= a \dot{a}\, r^2 \sin^2\theta$$

$$\Gamma^0_{11} = \frac{a\dot{a}}{1-kr^2}, \quad \Gamma^0_{22} = a\dot{a}\, r^2, \quad \Gamma^0_{33} = a\dot{a}\, r^2 \sin^2\theta$$

$$\Gamma^1_{01} = \Gamma^1_{10} = \Gamma^2_{02} = \Gamma^2_{20} = \Gamma^3_{03} = \Gamma^3_{30} = \frac{\dot{a}}{a}$$

$$\Gamma^1_{22} = -r\left(1-kr^2\right) \qquad \Gamma^1_{33} = -r\left(1-kr^2\right)\sin^2\theta \qquad \Gamma^1_{11} = \frac{kr}{1-kr^2}$$

$$\Gamma^2_{12} = \Gamma^2_{21} = \Gamma^3_{13} = \Gamma^3_{31} = \frac{1}{r}$$

$$\Gamma^2_{33} = -\sin\theta\cos\theta \,, \quad \Gamma^3_{23} = \Gamma^3_{32} = \cot\theta$$

$$R_{00} = \cancelto{0}{\Gamma^\mu_{00,\mu}} - \Gamma^\mu_{0\mu,0} + \Gamma^\mu_{\sigma\mu}\Gamma^\sigma_{00} - \Gamma^\mu_{\sigma 0}\Gamma^\sigma_{0\mu}$$

$$= -\cancel{\Gamma^0_{00,0}} - \Gamma^1_{01,0} - \Gamma^2_{02,0} - \Gamma^3_{03,0}$$

$$-\Gamma^0_{\sigma 0}\cancel{\Gamma^\sigma_{00}} - \Gamma^1_{\sigma 0}\Gamma^\sigma_{01} - \Gamma^2_{\sigma 0}\Gamma^\sigma_{02} - \Gamma^3_{\sigma 0}\Gamma^\sigma_{03}$$

$$= -\Gamma^1 - \Gamma^2 \quad \Gamma^3$$

$$- \Gamma^{1}_{01,0} - \Gamma^{2}_{02,0} - \Gamma^{3}_{03,0}$$

$$- \cancel{\Gamma^{1}_{00}} \Gamma^{0}_{01} - \Gamma^{1}_{10} \Gamma^{1}_{01} - \cancel{\Gamma^{1}_{20}} \Gamma^{2}_{01} - \cancel{\Gamma^{1}_{30}} \Gamma^{3}_{01}$$

$$- \cancel{\Gamma^{2}_{00}} \Gamma^{0}_{02} - \cancel{\Gamma^{2}_{10}} \Gamma^{1}_{02} - \Gamma^{2}_{20} \Gamma^{2}_{02} - \Gamma^{2}_{30} \cancel{\Gamma^{3}_{02}}$$

$$- \cancel{\Gamma^{3}_{00}} \Gamma^{0}_{03} - \Gamma^{3}_{10} \cancel{\Gamma^{1}_{03}} - \Gamma^{3}_{20} \cancel{\Gamma^{2}_{03}} - \Gamma^{3}_{30} \Gamma^{3}_{03}$$

$$= - \underline{\Gamma^{1}_{01,0}} - \underline{\Gamma^{2}_{02,0}} - \underline{\Gamma^{3}_{03,0}}$$

$$- \underline{\Gamma^{1}_{10} \Gamma^{1}_{01}} - \underline{\Gamma^{2}_{20}} \, \underline{\Gamma^{2}_{02}} - \underline{\Gamma^{3}_{30} \Gamma^{3}_{03}}$$

$$= -3 \frac{d}{dt} \left(\frac{\dot{a}}{a} \right) - 3 \left(\frac{\dot{a}}{a} \right)^2$$

$$= -3\left(\frac{\ddot{a}a - \dot{a}\dot{a}}{a^2} - \right) - 3\left(\frac{\dot{a}}{a}\right)$$

$$= -3\frac{\ddot{a}}{a} + \cancel{3\frac{\dot{a}^2}{a^2}} - 3\cancel{\left(\frac{\dot{a}}{a}\right)^2}$$

$$R_{00} = -3\frac{\ddot{a}}{a}$$

$$R_{11} = \frac{a\ddot{a} + 2\dot{a}^2 + 2k}{1 - kr^2}$$

$$R_{22} = r^2\left(a\ddot{a} + 2\dot{a}^2 + 2k\right)$$

$$R_{33} = r^2\sin^2\theta\left(a\ddot{a} + 2\dot{a}^2 + 2k\right)$$

Escalar de Ricci.

$$R = g^{\mu\nu}R_{\mu\nu}$$

$$R = g^{00}R_{00} + g^{11}R_{11} + g^{22}R_{22} + g^{33}R_{33}$$

$$= 3\frac{\ddot{a}}{a} + \frac{(1-\cancel{k}r^2)}{a^2}\,\frac{(a\ddot{a} + 2\dot{a}^2 + 2k)}{\cancel{1}-kr^2}$$

$$+ \frac{1}{a^2 \cancel{r^2}}\cancel{r^2}(a\ddot{a} + 2\dot{a}^2 + 2k) + \frac{1}{a^2\cancel{r^2}\cancel{\sin^2\theta}}\cancel{r^2}\cancel{\sin^2\theta}(a\ddot{a} + 2\dot{a}^2 + 2k)$$

$$= 3\frac{\ddot{a}}{a} + \frac{3(a\ddot{a} + 2\dot{a}^2 + 2k)}{a^2}$$

$$R = \frac{6}{a^2}\left(a\ddot{a} + \dot{a}^2 + k\right)$$

Einsten tensor:

$$G_{00} = R_{00} - \frac{1}{2}g_{00}R$$

$$= -3\frac{\ddot{a}}{a} + \frac{1}{2}\frac{6}{a^2}\left(a\ddot{a} + \dot{a}^2 + k\right)$$

$$= -3\frac{\ddot{a}}{a} + \frac{3\ddot{a}}{a} + \frac{3}{a^2}\left(\dot{a}^2 + k\right)$$

$$G_{00} = \frac{3}{a^2}\left(\dot{a}^2 + k\right)$$

$$G_{11} = -\frac{1}{1-kr^2}\left(2a\ddot{a} + \dot{a}^2 + k\right)$$

$$G_{22} = -r^2\left(2a\ddot{a} + \dot{a}^2 + k\right)$$

$$G_{33} = -r^2\sin^2\theta\left(2a\ddot{a} + \dot{a}^2 + k\right)$$

Ecuaciones de campo de Einstein.

$$G_{\mu\nu} = 8\pi T_{\mu\nu}$$

$$T_{\mu\nu} = (\rho + P)\, u_\mu u_\nu + P\, g_{\mu\nu}$$

$$u^\alpha u_\alpha = -1$$

$$= g_{00} (u^0)^2$$

$$= -(u^0)^2 \Rightarrow u^0 = 1 \;\wedge\; u_0 = g_{00} u^0 = -1$$

* $$T_{00} = (\rho + P) + P\, g_{00}$$

$$= \rho + \cancel{P} \cancel{-} P$$

$$= \rho$$

* $$T_{11} = P\, g_{11}$$

$$= \frac{a^2}{1 - kr^2} P$$

$$* \; T_{22} = p\, g_{22}$$

$$= a^2 r^2 p$$

$$* \; T_{33} = p\, g_{33}$$

$$= a^2 r^2 \sin^2\theta \; p$$

The Einstein Equations:

$$G_{00} = 8\pi T_{00}$$

$$\frac{3}{a^2}\left(\dot{a}^2 + k\right) = 8\pi \rho$$

$$\Rightarrow \left(\frac{\dot{a}}{a}\right)^2 = 8\pi \rho - \frac{k}{a^2}$$

Ecuación de Friedmann

* $\mu = ii$

$$G_{ii} = 8\pi T_{ii}$$

Para $i = 1$

$$G_{11} = 8\pi T_{11}$$

$$-\frac{1}{(1-kr^2)}\left(2a\ddot{a} + \dot{a}^2 + k\right) = 8\pi \left(-\frac{a^2}{1-kr^2}\right) p$$

$$2\frac{\ddot{a}}{a} + \left(\frac{\dot{a}}{a}\right)^2 + \frac{k}{a^2} = 8\pi p$$

$$2\frac{\ddot{a}}{a} + \frac{8\pi}{3}\rho - \frac{K}{a^2} + \frac{K}{a^2} = -8\pi P$$

$$2\frac{\ddot{a}}{a} = -8\pi\left(P + \frac{\rho}{3}\right)$$

$$\Rightarrow \frac{\ddot{a}}{a} = -\frac{4\pi}{3}\left(\rho + 3P\right)$$ Ecuación de la aceleración

Conservación del tensor energía-momento

$$T^{\mu\nu}{}_{;\nu} = T^{\mu\nu}{}_{,\nu} + \Gamma^{\mu}_{\sigma\nu} T^{\sigma\nu} + \Gamma^{\nu}_{\sigma\nu} T^{\sigma\mu}$$

* $\mu = 0$

$$T^{0\nu} = T^{00}{}_{,0} + T^{01}{}_{,1} + T^{02}{}_{,2} + T^{03}{}_{,3} = 0$$

$$
\begin{aligned}
&= T^{00}{}_{,0} + \Gamma^{0}{}_{\sigma 0} T^{\sigma 0} + \Gamma^{0}{}_{\sigma 0} T^{\sigma 0} \\
&+ \cancel{T^{\sigma 1}{}_{,1}} + \Gamma^{0}{}_{\sigma 1} T^{\sigma 1} + \Gamma^{1}{}_{\sigma 1} T^{\sigma 0} \\
&+ \cancel{T^{\sigma 2}{}_{,2}} + \Gamma^{0}{}_{\sigma 2} T^{\sigma 2} + \Gamma^{2}{}_{\sigma 2} T^{\sigma 0} \\
&+ \cancel{T^{\sigma 3}{}_{,3}} + \Gamma^{0}{}_{\sigma 3} T^{\sigma 3} + \Gamma^{3}{}_{\sigma 3} T^{\sigma 0}
\end{aligned}
$$

$$
\begin{aligned}
0 &= T^{00}{}_{,0} \\
&+ \Gamma^{0}{}_{11} T^{11} + \Gamma^{1}{}_{01} T^{00} \\
&+ \Gamma^{0}{}_{22} T^{22} + \Gamma^{2}{}_{02} T^{00}
\end{aligned}
$$

$$+ \Gamma^{0}_{33} T^{33} + \Gamma^{3}_{03} T^{00}$$

$$0 = \frac{d}{dt}(\rho) + \frac{3\dot{a}}{a}\rho + \frac{a\dot{a}}{1-kr^2} T^{11} + a\dot{a}r^2 T^{22} + a\dot{a}r^2 \sin^2\theta \, T^{33}$$

$$= \dot{\rho} + 3\frac{\dot{a}}{a}\rho + \frac{a\dot{a}}{1-\cancel{kr^2}} g^{11} g^{11} T_{11} + \cancel{a}\dot{a}\cancel{r^2} \cancel{g^{22}} g^{22} T_{22} + \cancel{a}\dot{a}\cancel{r^2}\cancel{\sin^2\theta} \cancel{g^{33}} g^{33} T_{33}$$

$$= \dot{\rho} + 3\frac{\dot{a}}{a}\rho + \frac{a\dot{a}}{a^2} g^{11} g_{11} P + \frac{a\dot{a}}{a^2} g^{22} g_{22} P + \frac{a\dot{a}}{a^2} g^{33} g_{33} P$$

$$0 = \dot{\rho} + 3\frac{\dot{a}}{a}\rho + 3\frac{\dot{a}}{a}P$$

$$P = w\rho$$

$$0 = \dot{\rho} + 3\frac{\dot{a}}{a}\rho + 3\frac{\dot{a}}{a}(w\rho)$$

$$\dot{\rho} = -3\frac{\dot{a}}{a}\rho(1+\omega)$$

$$\frac{\dot{\rho}}{\rho} = -3(1+\omega)\frac{\dot{a}}{a}$$

$$\ln\rho = -3(1+\omega)\ln a + \text{cte}$$

$$\Rightarrow \rho \propto a^{-3(1+\omega)}$$

Radiación: $\omega = 1/3 \rightarrow \rho \propto a^{-4}$

Materia: $\omega = 0 \rightarrow \rho \propto a^{-3}$

Vacío: $\omega = -1 \rightarrow \rho \rightarrow \rho_0$

$* \; \mu = i$

$$T^{i\nu}{}_{;\nu} = 0$$

$$T^{2\nu}{}_{;\nu} = 0$$

$$T^{3\nu}{}_{;\nu} = 0$$

Solución a la ecuación de Friedmann:

$$K = 0$$

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi}{3}\rho$$

$$\dot{a} = \sqrt{\frac{8\pi}{3}\rho}\; a$$

$$V\ \frac{}{3}$$

$$\frac{da}{a} = \sqrt{\frac{8\pi\rho}{3}}\,dt$$

$$\Rightarrow \quad \frac{da}{a} \propto \sqrt{\rho}\,dt$$

Radiación: $\rho \propto a^{-4}$

$$\frac{da}{a} \propto a^{-2}\,dt$$

$$a\,da \propto dt \Rightarrow a^2 \propto t$$

$$a \propto t^{1/2}$$

Materia: $\rho \propto a^{-3}$

$$\frac{da}{a} \propto a^{-3/2}\, dt$$

$$\sqrt{a}\, da \propto dt \Rightarrow a^{3/2} \propto t$$

$$a \propto t^{2/3}$$

Vacío: $\rho \propto \rho_0$

$$\frac{da}{a} \propto \sqrt{\rho_0}\, dt$$

$$\ln a \propto \sqrt{\rho_0}\, dt \Rightarrow a \propto e^{\sqrt{\rho_0}\, t}$$

Expansión desacelerada:

Radiación: $a \propto t^{1/2} \rightarrow \ddot{a} < 0$

Materia: $a \propto t^{2/3} \rightarrow \ddot{a} < 0$

Expansión acelerada:

Vacío: $a \propto e^{\sqrt{\rho_0}\, t} \rightarrow \ddot{a} > 0$