Física de Partículas

Desintegraciones y Dispersiones

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Latin American alliance for
Capacity buildiNG in Advanced physics
LA-CoNGA physics

Cofinanciado por el
programa Erasmus+
de la Unión Europea

Experimental probes

- Bound states: Static properties such as mass, spin, parity, magnetic moments

- Particle decays: Allowed and forbidden decays / Conservation laws

- Particle scattering: Production of new massive particles / Study of particle interaction cross sections / High energies to study short distances

Force	Typical Lifetime [s]	Typical cross-section [mb]
Strong	10^{-23}	10
Electromagnetic	10^{-20}	10^{-2}
Weak	10^{-8}	10^{-13}

Fermi's Golden Rule

- Particle decays and particle scattering are transitions between quantum mechanical states
- In QM the transition rate between states i and j is:

$$\Gamma_{fi} = 2\pi \, |T_{fi}|^2 \rho(E_i)$$

where T_{fi} is the transition matrix element and ρ is the density of states

Decay rates

- Lifetime of a particle (average or mean)

- Decay rate (Γ): probability per unit time that the particle of interest will decay

- If we had $N(t)$ particles, $N\Gamma dt$ particles would decay in the next instant dt

$$\mathrm{d}N = -\Gamma N \,\mathrm{d}t$$

- It follows that

$$N(t) = N(0)e^{-\Gamma t}$$

- We can see that the mean lifetime:

$$\tau = \frac{1}{\Gamma}$$

Decay rates

- Lifetime of a particle (average or mean)

- Decay rate (Γ): probability per unit time that the particle of interest will decay

- Rate of decays

$$\frac{dN}{dt} = -\Gamma N(t)$$

- Activity

$$A(t) = \left|\frac{dN}{dt}\right| = \Gamma N(t)$$

Decay rates

- Particles can decay in several ways (decay modes, channels)
- The total decay rate is the sum of the individual decay rates

$$\Gamma = \sum_j \Gamma_j.$$

- Branching ratios: relative frequency of a particular decay mode:

$$BR(j) = \frac{\Gamma_j}{\Gamma}$$

- Decaying states do not correspond to a single energy – they have a width:

$$\Delta E\,\tau \sim \hbar \quad \xrightarrow{\text{yields}} \quad \Delta E \sim \frac{\hbar}{\tau} = \hbar\Gamma$$

Decaying states in QM

- For a decaying state the probability density must decay exponentially: $\psi(t) = \psi(0)\mathrm{e}^{-iE_0 t}\mathrm{e}^{-t/2\tau}$ $\quad |\psi(t)|^2 = |\psi(0)|^2\,\mathrm{e}^{-t/\tau}$
- The energies present in the wavefunction are given by the Fourier transform of $\psi(t)$:

$$f(\omega) = f(E) = \int_0^\infty \psi(t)\mathrm{e}^{iEt}\,\mathrm{d}t = \int_0^\infty \psi(0)\mathrm{e}^{-t(iE_0+\frac{1}{2\tau})}\mathrm{e}^{iEt}\,\mathrm{d}t$$

$$= \int_0^\infty \psi(0)\mathrm{e}^{-t(i(E_0-E)+\frac{1}{2\tau})}\,\mathrm{d}t = \frac{i\psi(0)}{(E_0-E)-\frac{i}{2\tau}}$$

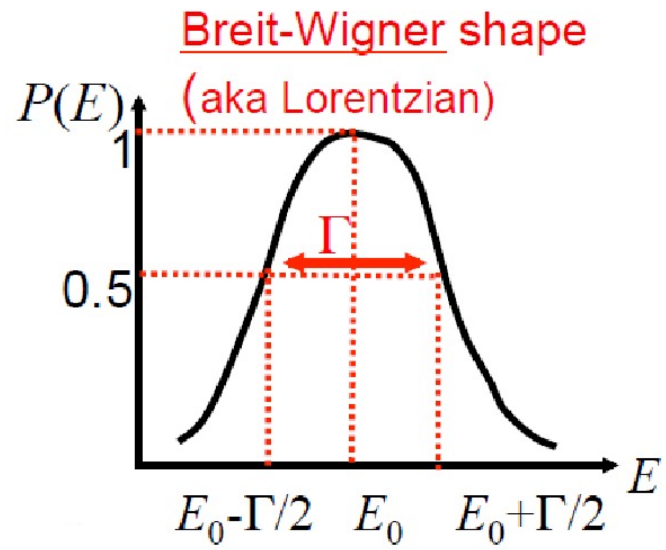
- So the probability of finding a state with energy E:

$$P(E) = |f(E)|^2 = \frac{|\psi(0)|^2}{(E_0-E)^2+\frac{1}{4\tau^2}}$$

- The probability density function for finding the particle with energy E is

$$p(E) \propto \frac{1}{(E_0 - E)^2 + \frac{\Gamma^2}{4}}$$

- E is the energy of the system
- E_0 is the characteristic rest-mass of the unstable particle
- The probability density function has a Lorentzian, peaked, line shape: ***Breit-Wigner***
- Full-width at half max (FWHM) of the peak equal to Γ: ***width***

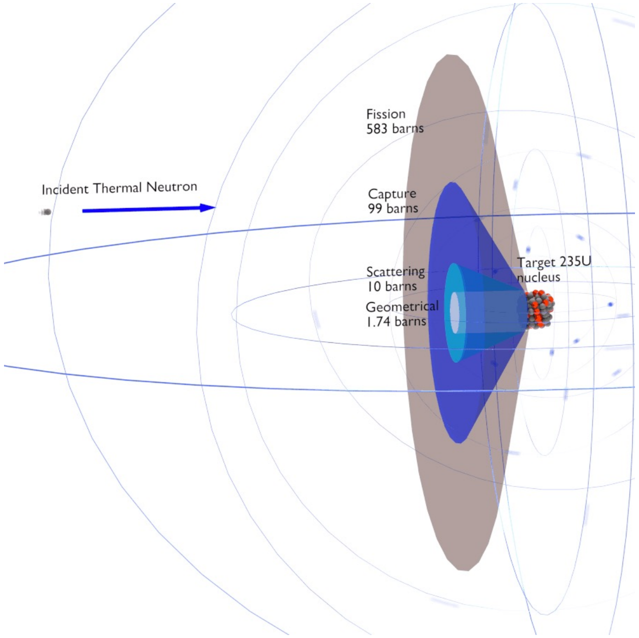


- Long-lived particles: narrow width, well defined energies

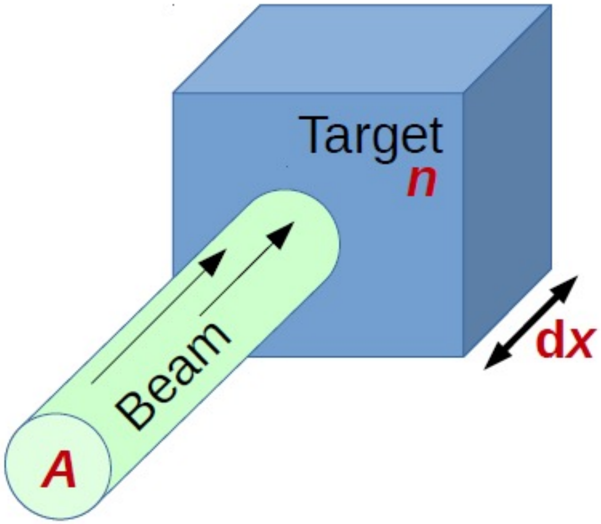
- ***Cross section***: "strength" of a particular interaction between two particles
- Effective target area presented to the incoming particle, units: barns ($1 \text{ barn} = 10^{-28}\text{m}^2$)
- Interaction rate per target particle:

$$\Gamma = \phi\sigma$$

- ϕ is the ***flux***: number of particles passing through unit area per second



- Consider a beam of N particles per unit time with area A
- The beam hits a target of n nuclei per unit volume and thickness dx



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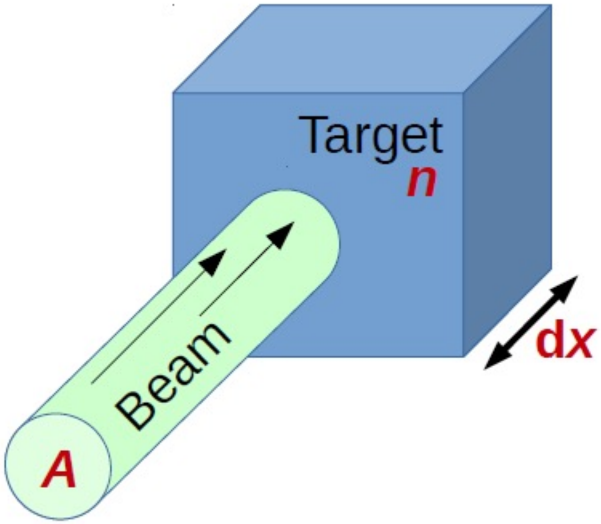

- Number of target particles in area A:

$$N_T = n \cdot A \cdot dx$$

- Effective area of interaction:

$$\sigma N_T = \sigma n A dx$$

- Incident flux:

$$\phi = N/A$$

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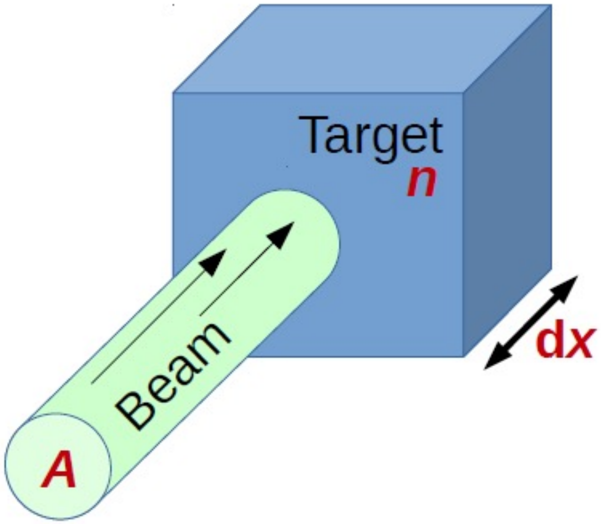

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- Number of particles scattered per unit time

$$-dN = \phi \sigma N_T = \frac{N}{A} \sigma n A dx$$

Scattering

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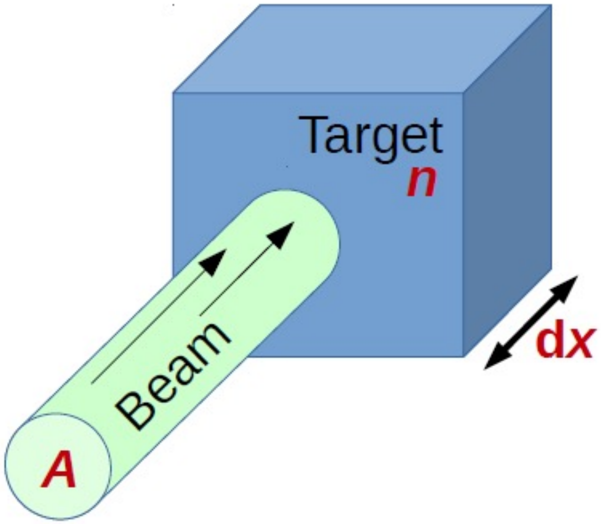

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$$N_T = n \cdot A \cdot dx$$

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- So the cross section is proportional to the scattering rate:

$$\sigma = \frac{-dN}{nNdx}$$

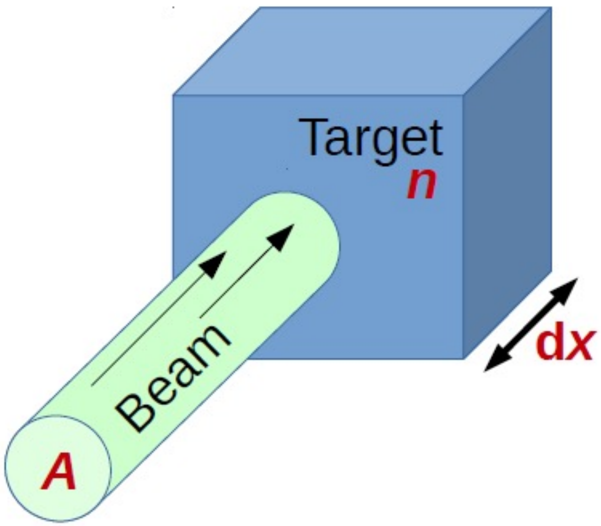
Beam attenuation in a target of thickness L:

- Thick target $\sigma n L \gg 1$:

$$\int_{N_0}^{N} -\frac{dN}{N} = \int_0^L \sigma n dx$$

$$N = N_0 e^{-\sigma n L}$$

the beam attenuates exponentially



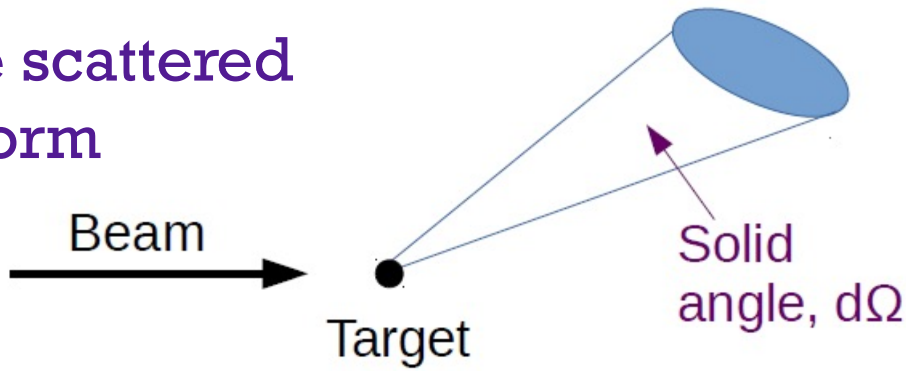
- Thin target $\sigma n L \ll 1$:

$$e^{-\sigma n L} \sim 1 - \sigma n L$$

$$N = N_0 (1 - \sigma n L)$$

- *Mean free path* between interactions: $1/\sigma n$ (also referred to as *interaction length*)

- The angular distribution of the scattered particles is not necessarily uniform



- Number of particles scattered per unit time into $d\Omega$ is

$$dN = d\sigma \phi N_T$$

- The differential cross-section:

$$\frac{d\sigma}{d\Omega} = \frac{dN}{d\Omega \phi N_T}$$

is the number of particles scattered per unit time and solid angle, divided by the incident flux and by the number of target nuclei defined by the beam area

Differential cross section

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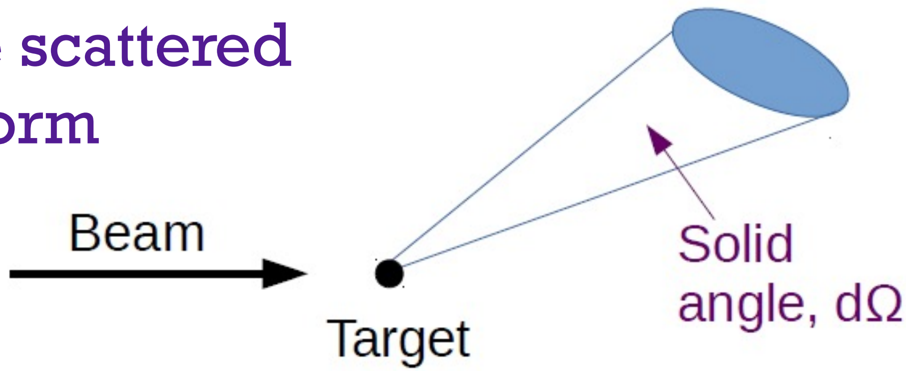

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- Most experiments do not cover 4π solid angle, and in general we measure $d\sigma/d\Omega$
- Angular distributions provide more information than the total cross-section about the mechanism of the interaction

Scattering in QM

- Consider a beam of particles scattering from a fixed potential $V(r)$
- The scattering rate is characterised by the interaction cross-section $\sigma = \Gamma/\phi$
- We can calculate the cross section using Fermi's golden rule

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- In first order perturbation theory, and using plane wave solutions:

$$\psi(\mathbf{x}, t) = Ae^{i(\mathbf{p}\cdot\mathbf{x}-Et)}$$

we need:

- Wave function normalisation
- Matrix element in perturbation theory
- Incident flux
- Density of states

- In first order perturbation theory, and using plane wave solutions:

$$\psi(\mathbf{x}, t) = Ae^{i(\mathbf{p}\cdot\mathbf{x} - Et)}$$

- Wave function normalisation: Normalise wave-functions to one particle in a box of side a

$$\int_0^a \int_0^a \int_0^a \psi^* \psi \, \mathrm{d}x \, \mathrm{d}y \, \mathrm{d}z = 1$$

$$A^2 = 1/a^3$$

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- The matrix element contains the physics of the interaction. In perturbation theory (first order):

$$T_{fi} = \langle f|\hat{H}|i\rangle$$

- Incident flux: consider a target of area A and a beam of particles with velocity v. Any incident particle within a volume vA will cross the target area every second

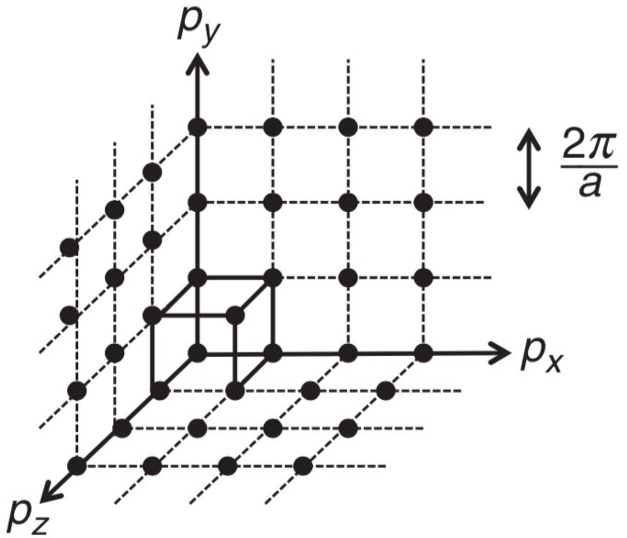
$$\phi = \frac{vA}{A} n = vn = \frac{v}{a^3}$$

- Density of states (or phase space): the normalisation of the wave function implies periodic boundary conditions, which implies the momentum components are quantised:

$$(p_x, p_y, p_z) = (n_x, n_y, n_z)\frac{2\pi}{a}$$

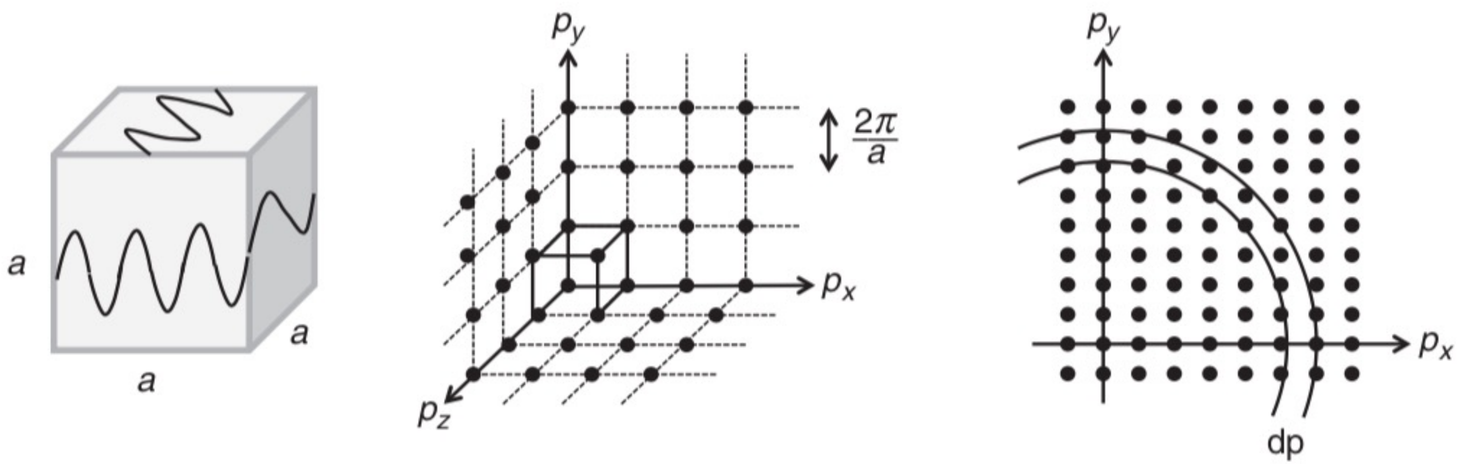
each state in momentum space occupies a cubic volume of

$$d^3\mathbf{p} = dp_x dp_y dp_z = \left(\frac{2\pi}{a}\right)^3 = \frac{(2\pi)^3}{V}$$



- Density of states (or phase space): the number of states dn with magnitude of momentum in the range $p \rightarrow p + dp$ is the volume (in momentum space) divided by the volume of a single state:

$$\mathrm{d}n = 4\pi \mathrm{p}^2 \mathrm{dp} \times \frac{V}{(2\pi)^3}$$



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and the density of states:

$$\rho(E) = \frac{\mathrm{d}n}{\mathrm{d}E} = \frac{\mathrm{d}n}{\mathrm{dp}}\left|\frac{\mathrm{dp}}{\mathrm{d}E}\right|$$

$$\frac{\mathrm{d}n}{\mathrm{dp}} = \frac{4\pi \mathrm{p}^2}{(2\pi)^3} V.$$

Putting everything together:

$$\sigma = \frac{\Gamma}{\phi} = \frac{2\pi T_{fi}^2 \rho(E)}{\phi}$$

$$T_{fi} = \langle f | \hat{H} | i \rangle$$

$$= \int \psi_f^* \hat{H} \psi_i \, d^3\vec{r}$$

$$= \int A e^{-i\vec{p}_f \cdot \vec{r}} V(\vec{r}) A e^{i\vec{p}_i \cdot \vec{r}} d^3\vec{r}$$

$$= A^2 \int e^{-i\vec{q}\cdot\vec{r}} V(\vec{r}) d^3\vec{r} \quad ; \quad \vec{q} = \vec{p}_f - \vec{p}_i$$

$\uparrow$

$a^3 = 1/V$

Putting everything together:

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$$|T_{fi}|^2 = \frac{1}{V^2} \left| \int e^{-i\vec{q}\cdot\vec{r}} V(\vec{r})\, d^3\vec{r} \right|^2$$

$$\phi = \frac{v_0}{V} \quad ; \quad \rho(E) = \frac{dn}{dp} \left| \frac{dp}{dE} \right|$$

$$= d\Omega\, p^2 \frac{V}{(2\pi)^3} \frac{E}{p}$$

Putting everything together:

$$\sigma = \frac{\Gamma}{\phi} = \frac{2\pi T_{fi}^2 \rho(E)}{\phi}$$

$$d\sigma = 2\pi \frac{1}{V^2} \left| \int e^{-i\vec{q}\cdot\vec{r}} \cdot V(\vec{r})\, d^3\vec{r} \right|^2 d\Omega\, p^2 \frac{V}{(2\pi)^3} \frac{E}{p} \frac{V}{v_0}$$

$$\frac{d\sigma}{d\Omega} = \frac{1}{(2\pi)^2 v_0} \left| \int e^{-i\vec{q}\cdot\vec{r}} \cdot V(\vec{r})\, d^3\vec{r} \right|^2 p\, E$$

If $v \sim c \sim 1$, $p \sim E$, **Born approximation**:

$$\frac{d\sigma}{d\Omega} = \frac{E^2}{(2\pi)^2} \left| \int e^{-i\vec{q}\cdot\vec{r}} V(\vec{r}) d^3\vec{r} \right|^2$$

Yukawa potential

- Consider relativistic elastic scattering from a Yukawa potential

$$V(\vec{r}) = \frac{g\,\mathrm{e}^{-mr}}{r}$$

- Our matrix element then:

$$\begin{aligned}
\int \mathrm{e}^{-\mathrm{i}\vec{q}.\vec{r}}\, V(\vec{r})\,\mathrm{d}^3\vec{r} &= \int_0^\infty \int_0^{2\pi} \int_0^\pi V(r)\,\mathrm{e}^{iqr\cos\theta}\, r^2 \sin\theta\,\mathrm{d}\theta\,\mathrm{d}\phi\,\mathrm{d}r \\
&= \int_0^\infty \int_{-1}^{+1} 2\pi V(r)\,\mathrm{e}^{iqr\cos\theta}\, r^2\,\mathrm{d}(\cos\theta)\,\mathrm{d}r \\
&= \int_0^\infty 2\pi V(r)\left(\frac{\mathrm{e}^{iqr} - \mathrm{e}^{-iqr}}{iqr}\right) r^2\,\mathrm{d}r \\
&= \int_0^\infty 2\pi g \frac{\mathrm{e}^{-mr}}{r}\left(\frac{\mathrm{e}^{iqr} - \mathrm{e}^{-iqr}}{iqr}\right) r^2\,\mathrm{d}r \\
&= \int_0^\infty 2\pi g\,\mathrm{e}^{-mr}\left(\frac{\mathrm{e}^{iqr} - \mathrm{e}^{-iqr}}{iq}\right) \mathrm{d}r \\
&= \int_0^\infty \frac{2\pi g}{iq}\left(\mathrm{e}^{-r(m-iq)} - \mathrm{e}^{-r(m+iq)}\right) \mathrm{d}r \\
&= \frac{2\pi g}{iq}\left(\frac{1}{m-\mathrm{i}q} - \frac{1}{m+\mathrm{i}q}\right) = \frac{2\pi g}{iq}\frac{2iq}{m^2+q^2} \\
&= \frac{4\pi g}{m^2+q^2}
\end{aligned}$$

where we chose the z-axis along r: $\vec{q}.\vec{r} = -qr\cos\theta$

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where we chose the z-axis along r: $\vec{q}.\vec{r} = -qr\cos\theta$

$$|M_{if}|^2 = \frac{16\pi^2 g^2}{(m^2+q^2)^2}$$

Rutherford scattering

- Consider relativistic elastic scattering from a Coulomb potential

$$V(\vec{r}) = -\frac{Z\alpha}{r}$$

$$|M_{if}|^2 = \frac{16\pi^2 Z^2 \alpha^2}{q^4}$$

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$$\vec{q} = \overrightarrow{p_f} - \overrightarrow{p_i}$$

$$|\vec{q}|^2 = 2|\vec{p}|^2(1 - \cos\theta) = 4E^2 \sin^2\frac{\theta}{2}$$

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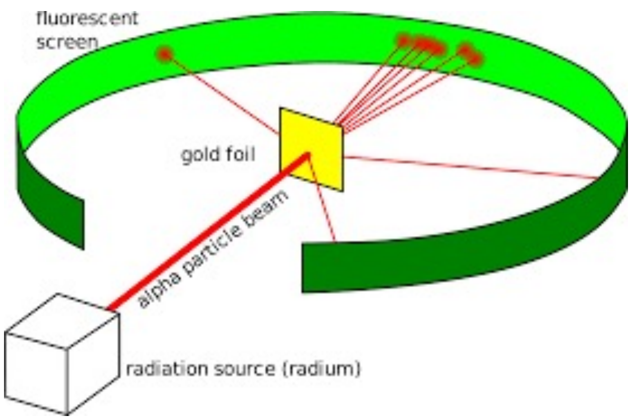
the differential cross section then:

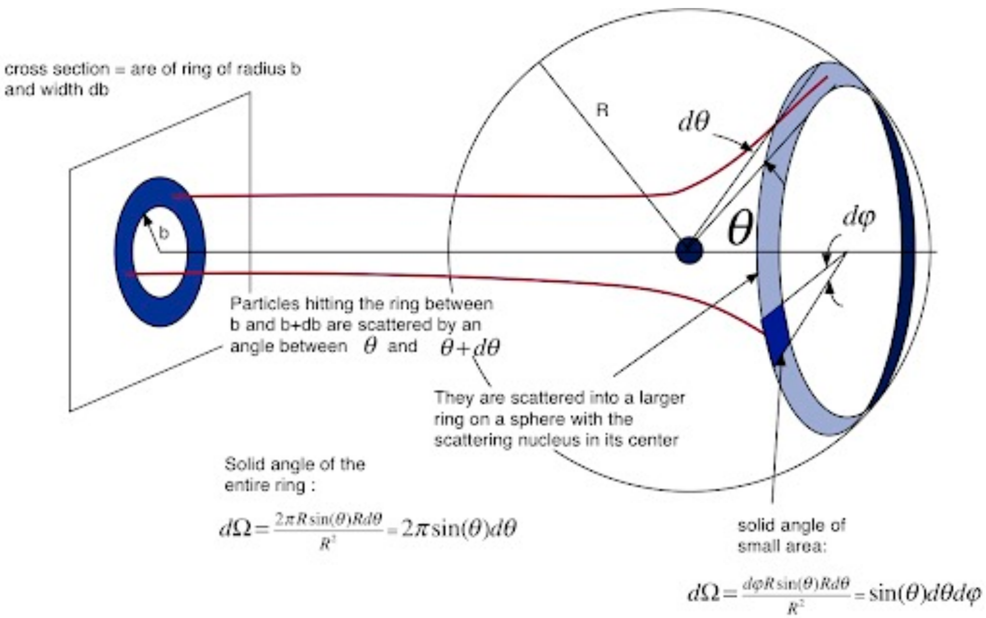
$$\frac{d\sigma}{d\Omega} = \frac{E^2}{(2\pi)^2}|\mathcal{M}|^2 = \frac{E^2}{(2\pi)^2}\frac{16\pi^2 Z^2 \alpha^2}{16E^4 \sin^4\frac{\theta}{2}}$$

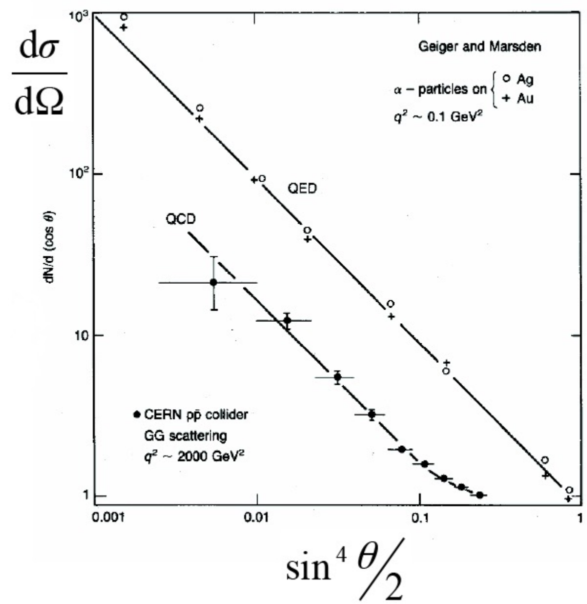
$$\frac{d\sigma}{d\Omega} = \frac{Z^2\alpha^2}{4E^2\sin^4\frac{\theta}{2}}$$

Geiger-Marsden experiment

- *Fixed target* experiment
- Alpha particles shot at a target
- Metal foil as target (Au and Ag)



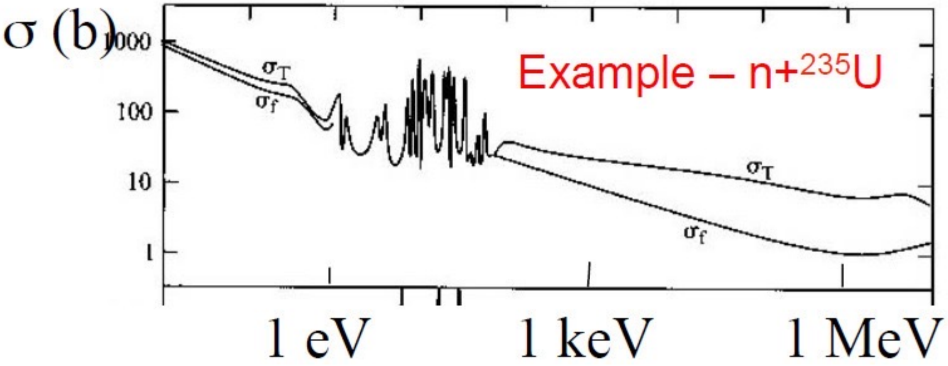




Resonant scattering

- Some particle interactions occur via an intermediate resonant state which then decays

$$a + b \rightarrow O \rightarrow c + d$$



- The matrix element is given by second order perturbation theory

$$T_{fi} = \langle f|V|i\rangle + \sum_{j\neq i} \frac{\langle f|V|j\rangle\langle j|V|i\rangle}{E_i - E_j}$$

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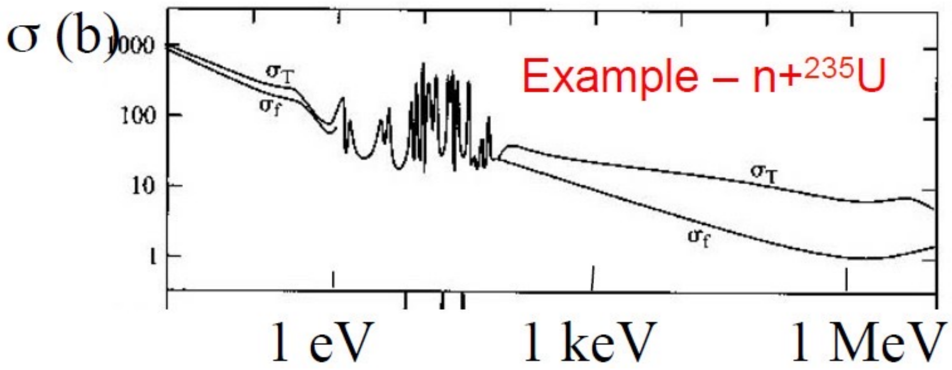

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- Two stage picture:

Production: $a + b \rightarrow O$ Decay: $O \rightarrow c + d$

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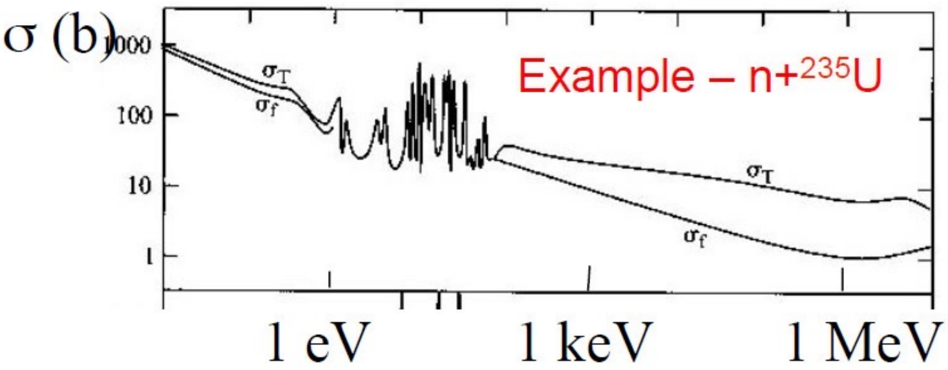

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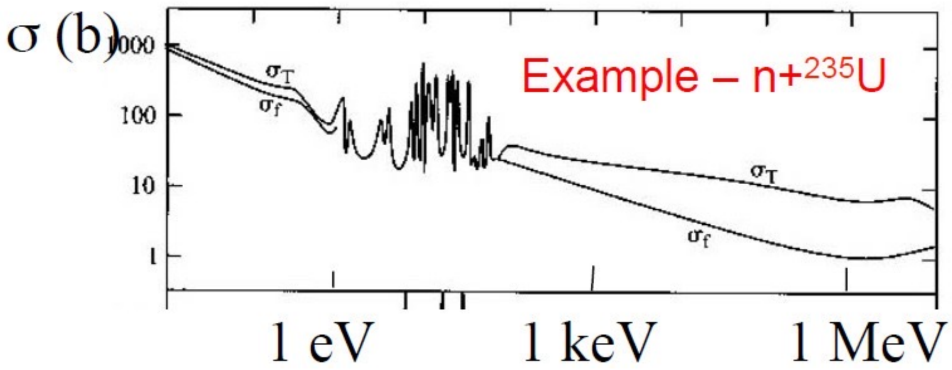

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- Near the *resonance* ($E \sim E_0 \sim m_O$) – 2nd order effects are large
- To account for the fact that O is unstable:

$$\psi \propto e^{-imt} \quad \to \quad \psi \propto e^{-imt} e^{-\Gamma t/2}$$

$$m \to m - i\Gamma/2$$

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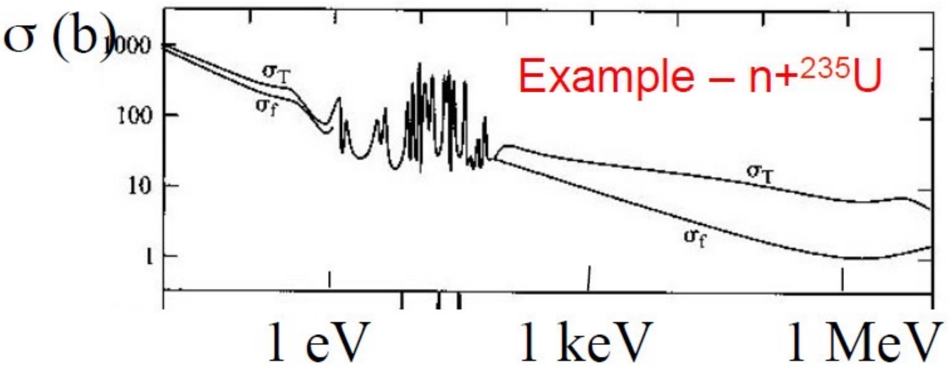

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$$T_{fi} = \langle f|V|i\rangle + \sum_{j\neq i} \frac{\langle f|V|j\rangle\langle j|V|i\rangle}{E_i - E_j}$$

- The matrix element squared is then:

$$|T_{fi}|^2 = \frac{|T_{fO}|^2 |T_{Oi}|^2}{(E - E_O)^2 + \frac{\Gamma^2}{4}}$$

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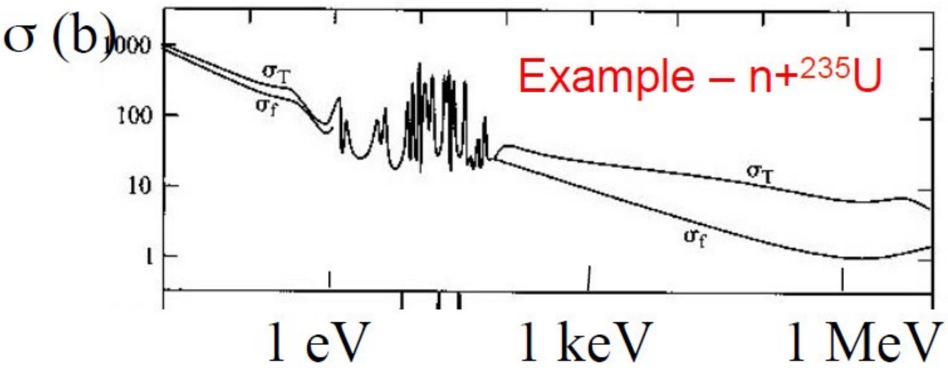

- So we have for the cross section:

$$\sigma = \frac{\pi}{p_i^2} \frac{\Gamma_{O\to i}\Gamma_{O\to f}}{(E - E_O)^2 + \frac{\Gamma^2}{4}}$$

this is the **Breit-Wigner** cross section

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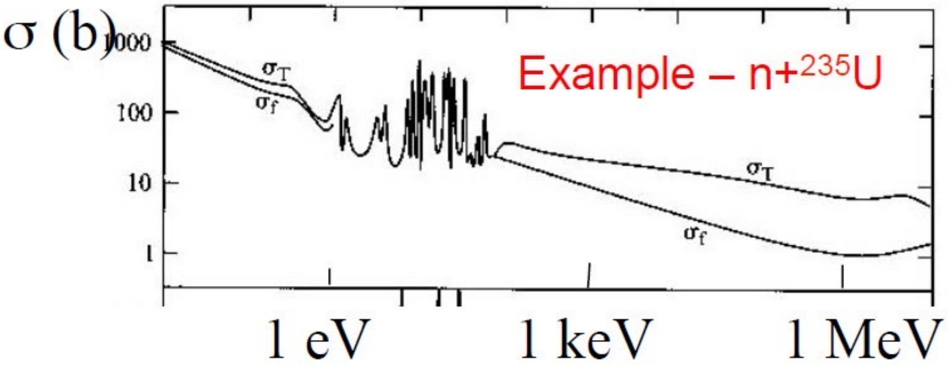
- Some particle interactions occur via an intermediate resonant state which then decays

$$a + b \to O \to c + d$$



- So we have for the cross section:

$$\sigma = \frac{\pi}{p_i^2} \frac{\Gamma_{O\to i}\Gamma_{O\to f}}{(E - E_O)^2 + \frac{\Gamma^2}{4}}$$

this is the **Breit-Wigner** cross section

- p_i^2 is calculated in the centre-of-mass frame
- E is the centre-of-mass energy,
- E_O is the rest mass of the resonance
- $\Gamma_{O\to x}$ are partial widths and Γ the full width of the resonance

Resonant scattering

- We should also include information about spin:

$$\sigma = \frac{g\pi}{p_i^2} \frac{\Gamma_{O \to i} \Gamma_{O \to f}}{(E - E_O)^2 + \frac{\Gamma^2}{4}}$$

with:

$$g = \frac{2J_O + 1}{(2J_a + 1)(2J_b + 1)}$$

is the ratio of the number of spin states for the resonant state to the total number of spin states for the $a + b$ system

- It is the probability that $a + b$ collide in the correct spin state to form the resonance O

Resonant scattering

- We can use measurements of cross sections to infer other information

- Total cross section:

$$\sigma_{tot} = \sum_f \sigma(i \to f)$$

$$\sigma_{tot} = \frac{g\pi}{p_i^2} \frac{\Gamma_{O\to i}\Gamma}{(E - E_O)^2 + \frac{\Gamma^2}{4}}$$

Resonant scattering

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$$\sigma_{tot} = \sum_f \sigma(i \to f)$$

$$\sigma_{tot} = \frac{g\pi}{p_i^2} \frac{\Gamma_{O\to i}\Gamma}{(E - E_O)^2 + \frac{\Gamma^2}{4}}$$

- Elastic cross section:

$$\sigma_{el} = \sigma(i \to i)$$

$$\sigma = \frac{g\pi}{p_i^2} \frac{\Gamma_{O\to i}\Gamma_{O\to i}}{(E - E_O)^2 + \frac{\Gamma^2}{4}}$$

Resonant scattering

- We can use measurements of cross sections to infer other information

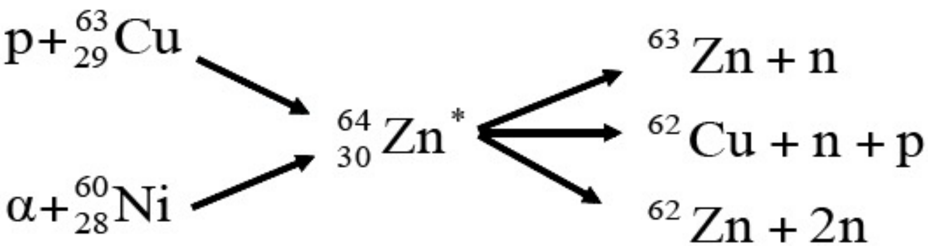
- On peak resonance ($E = E_0$)

$$\sigma_{peak} = \frac{g4\pi}{p_i^2}\frac{\Gamma_{O\to i}\Gamma_{O\to f}}{\Gamma^2}$$

$$\sigma_{peak-el} = \frac{g4\pi}{p_i^2}\frac{\Gamma_{O\to i}\Gamma_{O\to i}}{\Gamma^2} = \frac{g4\pi}{p_i^2}BR(i)^2$$

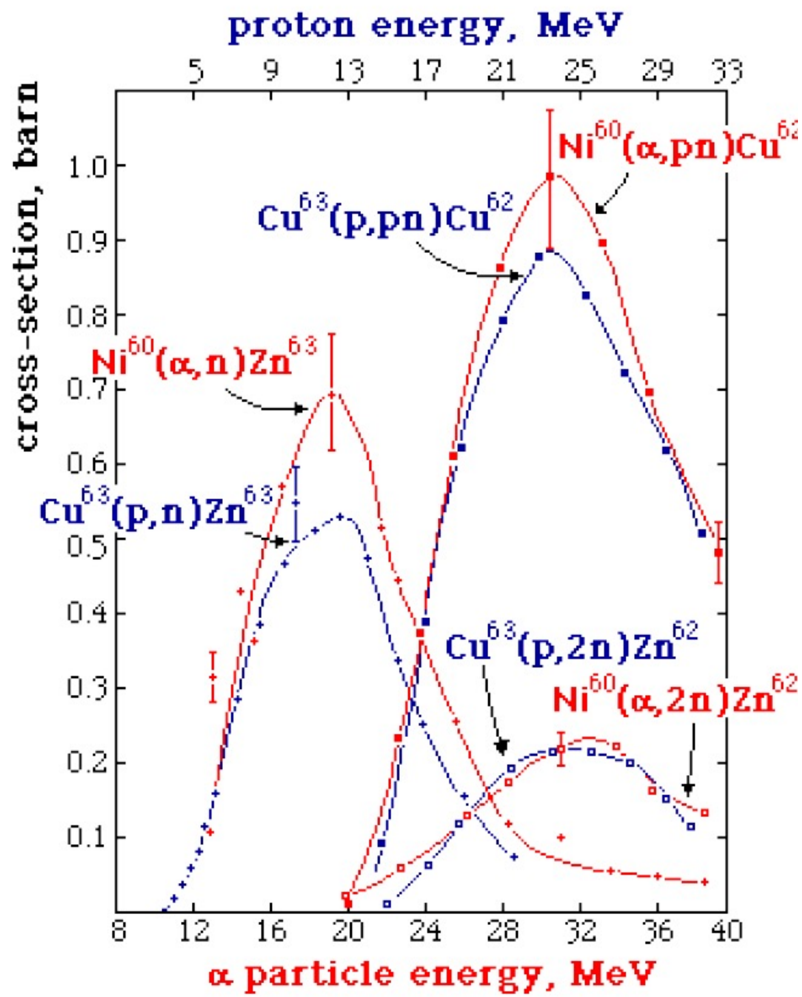
$$\sigma_{peak-tot} = \frac{g4\pi}{p_i^2}\frac{\Gamma_{O\to i}}{\Gamma} = \frac{g4\pi}{p_i^2}BR(i)$$

Resonances (nuclear physics)



- Production independent of decay
- We can see the 3 resonances from the 2 production mechanisms
- Notation in nuclear physics:

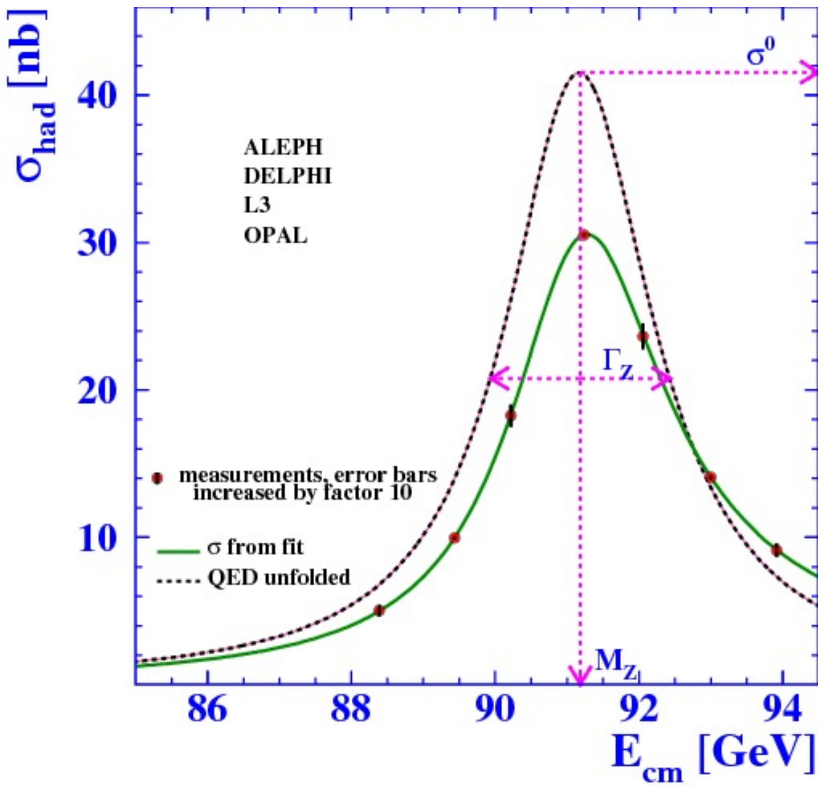
$$a + B \rightarrow c + D = B(a, c)D$$

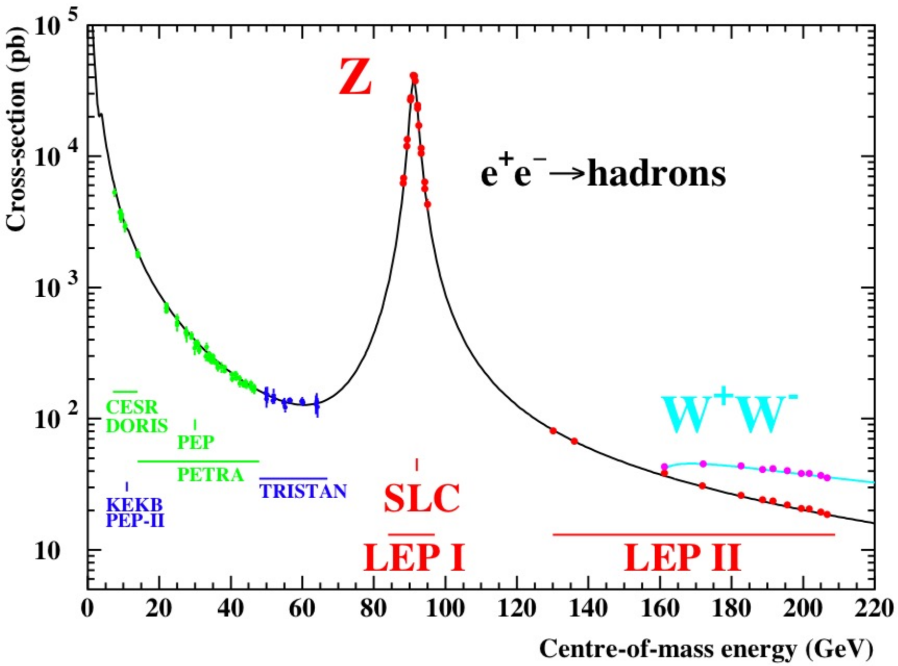


Resonances (particle physics)

- Z boson at LEP

$$m_Z = 91.1875 \pm 0.0021\,\text{GeV}$$





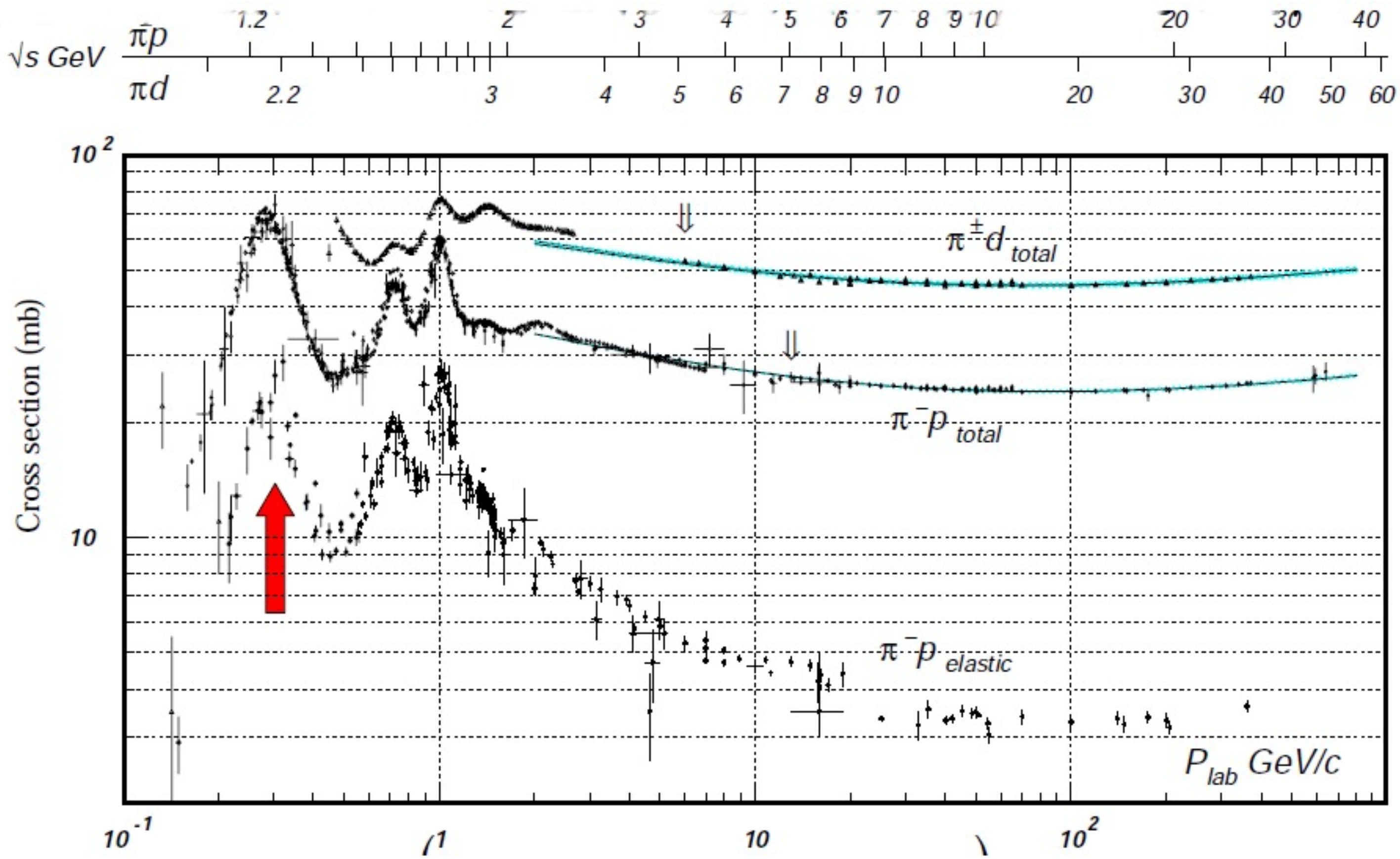
- Total decay width

$$\Gamma_Z = 2.4952 \pm 0.0023\,\text{GeV}$$

- Peak cross section

Example

- $\pi^- p$ **scattering: Resonance at** $p_\pi^{\mathrm{lab}} \sim 0.3\ \mathrm{GeV}$, $E_{\mathrm{cm}} = 1.25\ \mathrm{GeV}$. $\sigma_{peak-tot} = 72\mathrm{mb}$, $\sigma_{peak-el} = 28\mathrm{mb}$. **Find** g **and** J_0 ($J_p = \frac{1}{2}, J_\pi = 0$)

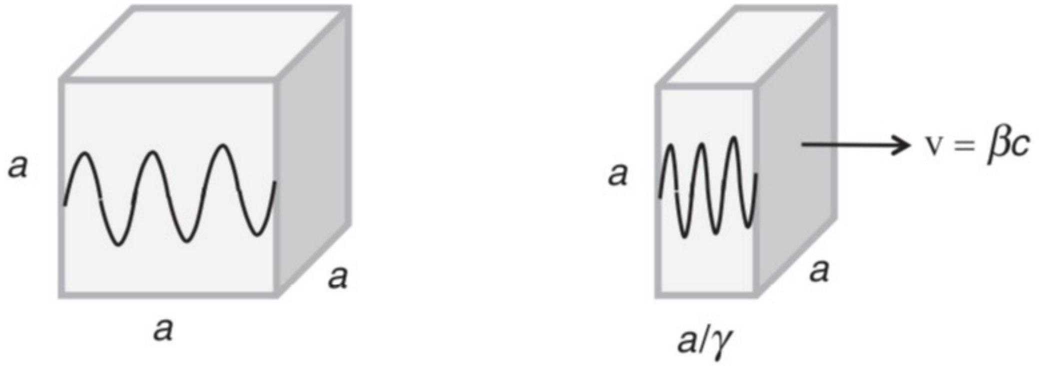


Fermi's Golden Rule (Relativistic)

- It was assumed before that the wave functions appearing on the transition matrix are normalised (1 particle per unit volume):

$$\int_0^a \int_0^a \int_0^a \psi^* \psi \, \mathrm{d}x \, \mathrm{d}y \, \mathrm{d}z = 1$$

which is not Lorentz invariant



Fermi's Golden Rule (Relativistic)

- The usual convention is to normalise to 2E particles per unit volume:

$$\int_V \psi'^* \psi' \, \mathrm{d}^3\mathbf{x} = 2E$$

in which case:

$$\psi' = (2E)^{1/2}\psi$$

- If we define a general Lorentz invariant matrix element :

$$\mathcal{M}_{fi} = \langle \psi_1' \psi_2' \cdots | \hat{H}' | \psi_a' \psi_b' \cdots \rangle$$

$$\mathcal{M}_{fi} = \langle \psi_1' \psi_2' \cdots | \hat{H}' | \psi_a' \psi_b' \cdots \rangle = (2E_1 \cdot 2E_2 \cdots 2E_a \cdot 2E_b \cdots)^{1/2} T_{fi}$$

Two body decay

- Consider a decay of the form $a \to 1 + 2$
- The NR-QM golden rule:

$$\Gamma_{fi} = 2\pi \int |T_{fi}|^2 \delta(E_a - E_1 - E_2)\, \mathrm{d}n$$

$$\Gamma_{fi} = (2\pi)^4 \int |T_{fi}|^2 \delta(E_a - E_1 - E_2)\delta^3(\mathbf{p}_a - \mathbf{p}_1 - \mathbf{p}_2) \frac{\mathrm{d}^3\mathbf{p}_1}{(2\pi)^3} \frac{\mathrm{d}^3\mathbf{p}_2}{(2\pi)^3}$$

- Using the Lorentz invariant matrix element:

$$\Gamma_{fi} = \frac{(2\pi)^4}{2E_a} \int |\mathcal{M}_{fi}|^2 \delta(E_a - E_1 - E_2)\delta^3(\mathbf{p}_a - \mathbf{p}_1 - \mathbf{p}_2) \frac{\mathrm{d}^3\mathbf{p}_1}{(2\pi)^3 2E_1} \frac{\mathrm{d}^3\mathbf{p}_2}{(2\pi)^3 2E_2}$$

with $|\mathcal{M}_{fi}|^2 = (2E_a 2E_1 2E_2)|T_{fi}|^2$

Two body decay

- Consider a decay of the form $a \to 1 + 2$

$$\Gamma_{fi} = \frac{(2\pi)^4}{2E_a} \int |\mathcal{M}_{fi}|^2 \delta(E_a - E_1 - E_2)\delta^3(\mathbf{p}_a - \mathbf{p}_1 - \mathbf{p}_2) \frac{\mathrm{d}^3\mathbf{p}_1}{(2\pi)^3 2E_1} \frac{\mathrm{d}^3\mathbf{p}_2}{(2\pi)^3 2E_2}$$

- The phase space integral $\mathrm{d}^3\mathbf{p}/(2\pi)^3$

  is replaced by $\frac{\mathrm{d}^3\mathbf{p}}{(2\pi)^3 2E}$

which is the Lorentz invariant phase space factor.

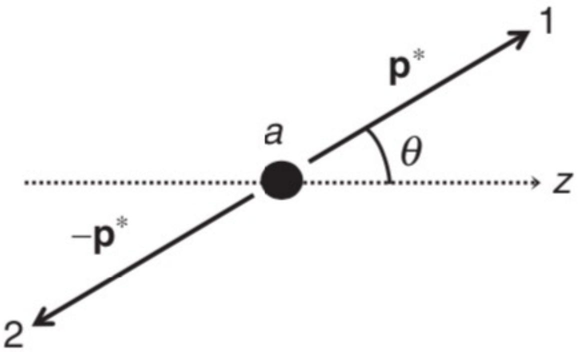
- This is the Lorentz invariant Golden rule for a two body decay

Two body decay

- Example: A particle of mass m (at rest) decays into two massless particles.

Two body decay

- General two body decay



$$\Gamma_{fi} = \frac{1}{8\pi^2 m_a} \int |\mathcal{M}_{fi}|^2 \delta(m_a - E_1 - E_2)\delta^3(\mathbf{p}_1 + \mathbf{p}_2) \frac{\mathrm{d}^3\mathbf{p}_1}{2E_1} \frac{\mathrm{d}^3\mathbf{p}_2}{2E_2}$$

$$\Gamma_{fi} = \frac{\mathrm{p}^*}{32\pi^2 m_a^2} \int |\mathcal{M}_{fi}|^2 \, \mathrm{d}\Omega$$

$$\mathrm{p}^* = \frac{1}{2m_a} \sqrt{\left[(m_a^2 - (m_1 + m_2)^2\right]\left[m_a^2 - (m_1 - m_2)^2\right]}$$

Decay rates

- In general

$$\Gamma = \frac{(2\pi)^4}{2E_a} \int |\mathcal{M}|^2 \delta^4(p_a - p_1 \ldots - p_n) \left(\frac{d^3\boldsymbol{p}_1}{(2\pi)^3 2E_1}\right) \left(\frac{d^3\boldsymbol{p}_2}{(2\pi)^3 2E_2}\right) \cdots \left(\frac{d^3\boldsymbol{p}_n}{(2\pi)^3 2E_n}\right)$$

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physics is contained in the matrix element

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4-momentum conservation

Decay rates

- In general

$$\Gamma = \frac{(2\pi)^4}{2E_a} \int |\mathcal{M}|^2 \delta^4(p_a - p_1 \dots - p_n) \left(\frac{d^3\boldsymbol{p}_1}{(2\pi)^3 2E_1}\right) \left(\frac{d^3\boldsymbol{p}_2}{(2\pi)^3 2E_2}\right) \dots \left(\frac{d^3\boldsymbol{p}_n}{(2\pi)^3 2E_n}\right)$$

physics is contained in the matrix element

4-momentum conservation

Lorentz invariant phase space factor

Cross sections

$$\sigma = \frac{\Gamma_{fi}}{(\mathrm{v}_a + \mathrm{v}_b)}$$

- Going back to the Golden rule:

$$\Gamma_{fi} = (2\pi)^4 \int |T_{fi}|^2 \delta(E_a + E_b - E_1 - E_2)\delta^3(\boldsymbol{p}_a + \boldsymbol{p}_b - \boldsymbol{p}_1 - \boldsymbol{p}_2)\left(\frac{d^3\boldsymbol{p}_1}{(2\pi)^3}\right)\left(\frac{d^3\boldsymbol{p}_2}{(2\pi)^3}\right)$$

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- Remember these factors are not Lorentz Invariant!

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- Remember these factors are not Lorentz Invariant!

$$\mathcal{M}_{fi} = \langle \psi_1' \psi_2' \cdots | \hat{H}' | \psi_a' \psi_b' \cdots \rangle = (2E_1 \cdot 2E_2 \cdots 2E_a \cdot 2E_b \cdots)^{1/2} T_{fi}$$

- We normalize to 2E particles per unit volume!

$$\frac{\mathrm{d}^3\mathbf{p}}{(2\pi)^3 2E}$$

Cross sections

$$\sigma = \frac{(2\pi)^{-2}}{4\,E_a E_b(\mathrm{v}_a + \mathrm{v}_b)} \int |\mathcal{M}_{fi}|^2 \delta(E_a + E_b - E_1 - E_2)\delta^3(\mathbf{p}_a + \mathbf{p}_b - \mathbf{p}_1 - \mathbf{p}_2)\frac{\mathrm{d}^3\mathbf{p}_1}{2E_1}\frac{\mathrm{d}^3\mathbf{p}_2}{2E_2}$$

- Which is now Lorentz Invariant
- Lorentz invariant flux factor: $4\,E_a E_b(\mathrm{v}_a + \mathrm{v}_b)$

$$F = 4\left[(p_a \cdot p_b)^2 - m_a^2 m_b^2\right]^{\frac{1}{2}}$$

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$$F = 4\left[(p_a \cdot p_b)^2 - m_a^2 m_b^2\right]^{\frac{1}{2}}$$

- Two particular cases

- centre-of-mass frame: $F = 4|p|\sqrt{s}$

- fixed-target (particle b at rest): $F = 4m_b|p_a|$

Two-body scattering in CM frame (*)

$$\sigma = \frac{(2\pi)^{-2}}{4\,E_a E_b(\mathrm{v}_a + \mathrm{v}_b)} \int |\mathcal{M}_{fi}|^2 \delta(E_a + E_b - E_1 - E_2)\delta^3(\mathbf{p}_a + \mathbf{p}_b - \mathbf{p}_1 - \mathbf{p}_2)\frac{\mathrm{d}^3\mathbf{p}_1}{2E_1}\frac{\mathrm{d}^3\mathbf{p}_2}{2E_2}$$

- **With:** $\mathbf{p}_a = -\mathbf{p}_b = \mathbf{p}_i^*$

$$\sqrt{s} = (E_a + E_b)$$

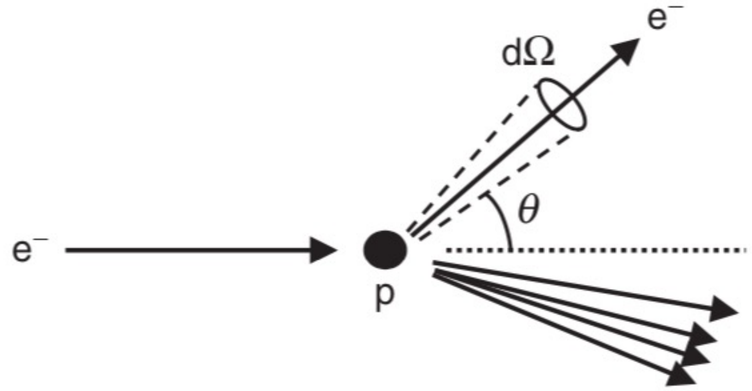
$$\sigma = \frac{1}{(2\pi)^2}\frac{1}{4\mathrm{p}_i^*\sqrt{s}} \int |\mathcal{M}_{fi}|^2 \delta\left(\sqrt{s} - E_1 - E_2\right)\delta^3(\mathbf{p}_1 + \mathbf{p}_2)\frac{\mathrm{d}^3\mathbf{p}_1}{2E_1}\frac{\mathrm{d}^3\mathbf{p}_2}{2E_2}$$

$$\sigma = \frac{1}{64\pi^2 s}\frac{\mathrm{p}_f^*}{\mathrm{p}_i^*}\int |\mathcal{M}_{fi}|^2 \mathrm{d}\Omega^*$$

where $\mathbf{p}_1 = -\mathbf{p}_2 = \mathbf{p}_f^*$

Differential cross section

- In some cases not only the total cross section is of interest



- here the angular distribution of the scattered electron provides crucial information

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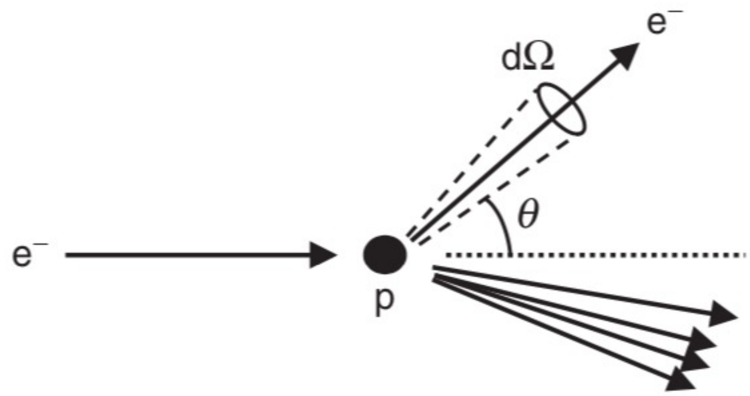

- here the angular distribution of the scattered electron provides crucial information
- Differential cross section:

$$\frac{d\sigma}{d\Omega} = \frac{\text{number of particles scattered into } d\Omega \text{ per unit time per target particle}}{\text{incident flux}}$$

Differential cross section

- Differential cross section:

$$\sigma = \int \frac{d\sigma}{d\Omega} d\Omega.$$

in general is not restricted to angular distributions

$$\frac{d\sigma}{dE} \qquad \frac{d^2\sigma}{dE\, d\Omega}$$

- Looking back at the two body scattering:

$$\frac{d\sigma}{d\Omega^*} = \frac{1}{64\pi^2 s} \frac{p_f^*}{p_i^*} |\mathcal{M}_{fi}|^2$$

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this works in the case where the C.M frame is the same as the lab. Frame (i.e. the pp collisions at the LHC)

- We need a Lorentz invariant form so it can be applied to any reference frame
- We introduce the Mandelstam variables:

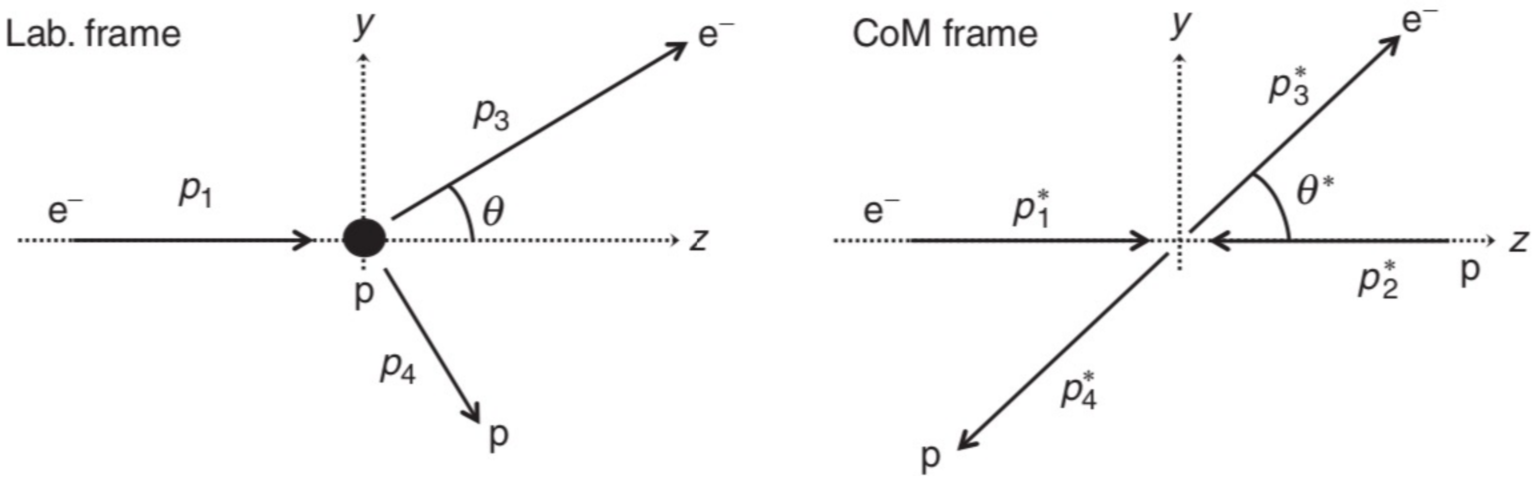
$$s = (p_1 + p_2)^2 = (p_3 + p_4)^2$$
$$t = (p_1 - p_3)^2 = (p_2 - p_4)^2$$
$$u = (p_1 - p_4)^2 = (p_2 - p_3)^2$$

since they are four-vector scalar products, they are Lorentz invariant

- Also:

$$s + u + t = m_1^2 + m_2^2 + m_3^2 + m_4^2$$

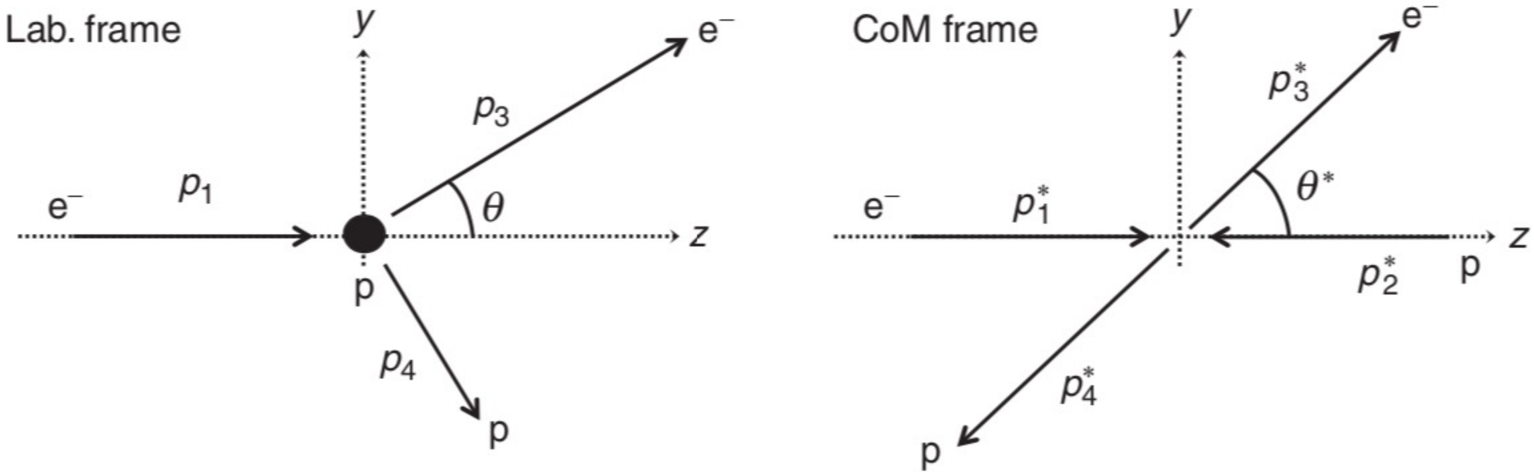
- If we take an ep elastic collision as example:



$$t = (p_1^* - p_3^*)^2 = p_1^{*2} + p_3^{*2} - 2p_1^* \cdot p_3^*$$
$$= m_1^2 + m_3^2 - 2(E_1^* E_3^* - \mathbf{p}_1^* \cdot \mathbf{p}_3^*)$$
$$= m_1^2 + m_3^2 - 2E_1^* E_3^* + 2\mathrm{p}_1^* \mathrm{p}_3^* \cos\theta^*$$

Differential cross section

- Here energies and momenta are fixed by energy and momentum conservation



$$t = (p_1^* - p_3^*)^2 = p_1^{*2} + p_3^{*2} - 2p_1^* \cdot p_3^*$$
$$= m_1^2 + m_3^2 - 2(E_1^* E_3^* - \mathbf{p}_1^* \cdot \mathbf{p}_3^*)$$
$$= m_1^2 + m_3^2 - 2E_1^* E_3^* + 2\mathrm{p}_1^* \mathrm{p}_3^* \cos\theta^*$$

$$\mathrm{d}t = 2\mathrm{p}_1^* \mathrm{p}_3^* \, \mathrm{d}(\cos\theta^*)$$

Differential cross section

- Going back to the differential cross section

$$d\sigma = \frac{1}{64\pi^2 s}\frac{p_f^*}{p_i^*}|\mathcal{M}_{fi}|^2 d\Omega^*$$

with

$$d\Omega^* \equiv d(\cos\theta^*)\, d\phi^* = \frac{dt\, d\phi^*}{2p_1^* p_3^*}$$

we get:

$$d\sigma = \frac{1}{128\pi^2 s\, p_i^{*2}}|\mathcal{M}_{fi}|^2 d\phi^* dt$$

and assuming the amplitude is independent of the azimuthal angle

$$\frac{d\sigma}{dt} = \frac{1}{64\pi s\, p_i^{*2}}|\mathcal{M}_{fi}|^2$$

Differential cross section

- Going back to the differential cross section

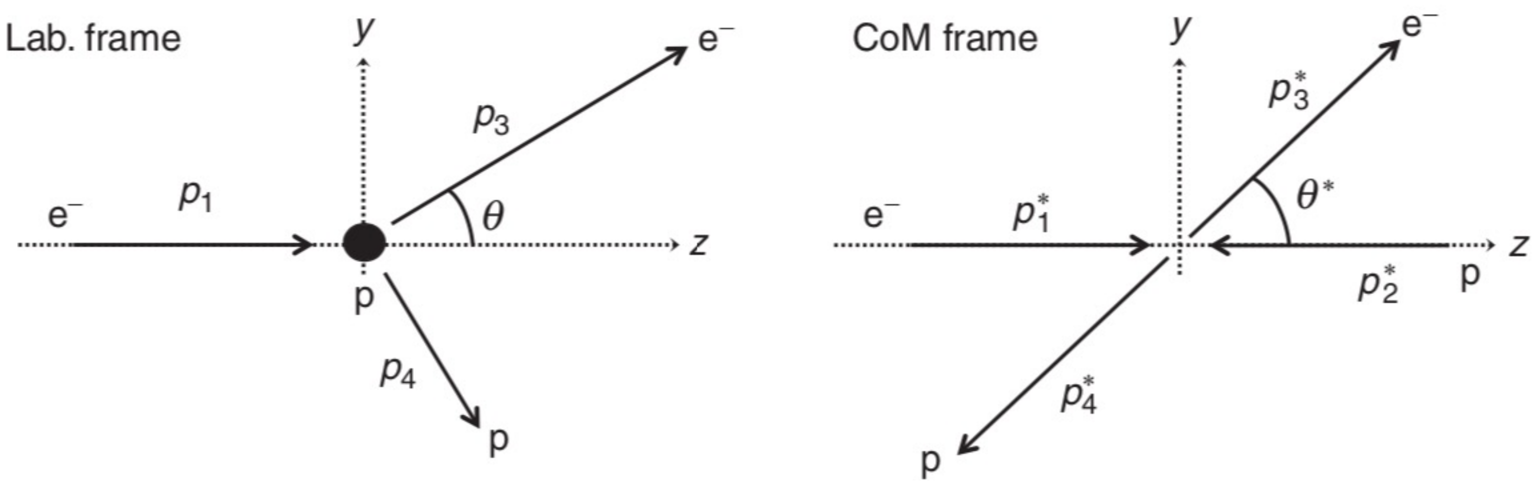
$$\frac{d\sigma}{dt} = \frac{1}{64\pi s\, p_i^{*2}} |\mathcal{M}_{fi}|^2$$

this is Lorentz invariant.

- Homework: prove that

$$p_i^{*2} = \frac{1}{4s}[s - (m_1 + m_2)^2][s - (m_1 - m_2)^2]$$

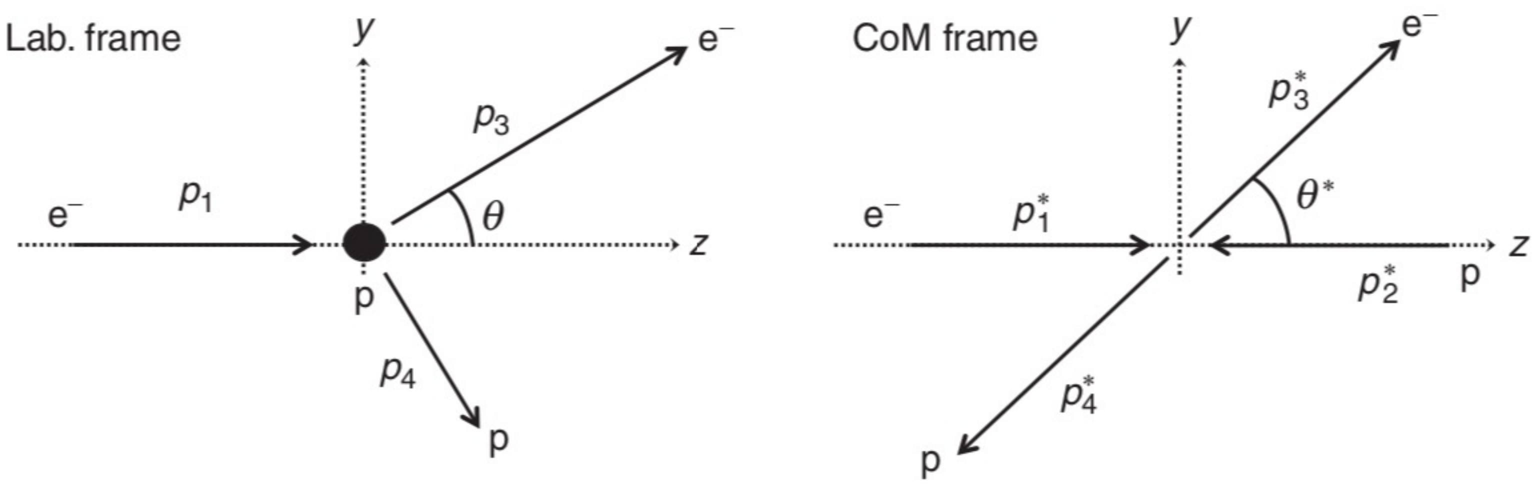
- Let's look at the lab frame now



In the limit where $E_e \approx p_e$

$$
\begin{aligned}
p_1 &\approx (E_1, 0, 0, E_1),\\
p_2 &= (m_{\mathrm{p}}, 0, 0, 0),\\
p_3 &\approx (E_3, 0, E_3 \sin\theta, E_3 \cos\theta),\\
p_4 &= (E_4, \mathbf{p}_4).
\end{aligned}
$$

- Let's look at the lab frame now



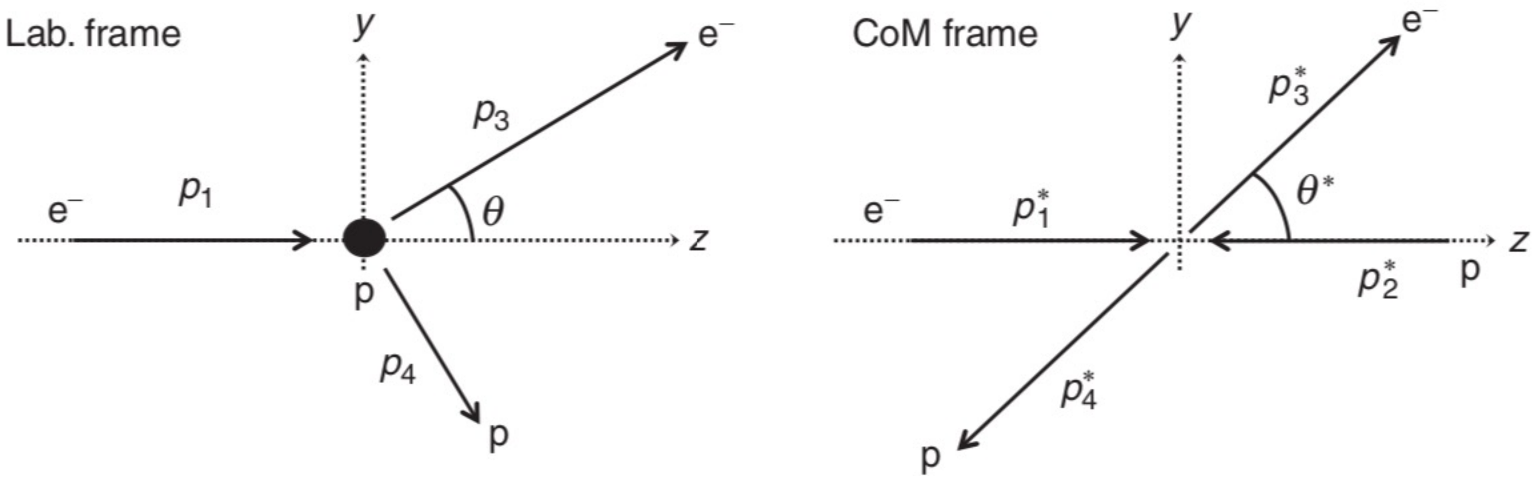
Since $m_e \ll m_p$ $\qquad p_i^{*2} \approx \frac{(s - m_p^2)^2}{4s}$

and

$$s = (p_1 + p_2)^2 = p_1^2 + p_2^2 + 2p_1 \cdot p_2 \approx m_p^2 + 2p_1 \cdot p_2$$
$$= m_p^2 + 2E_1 m_p,$$

Differential cross section

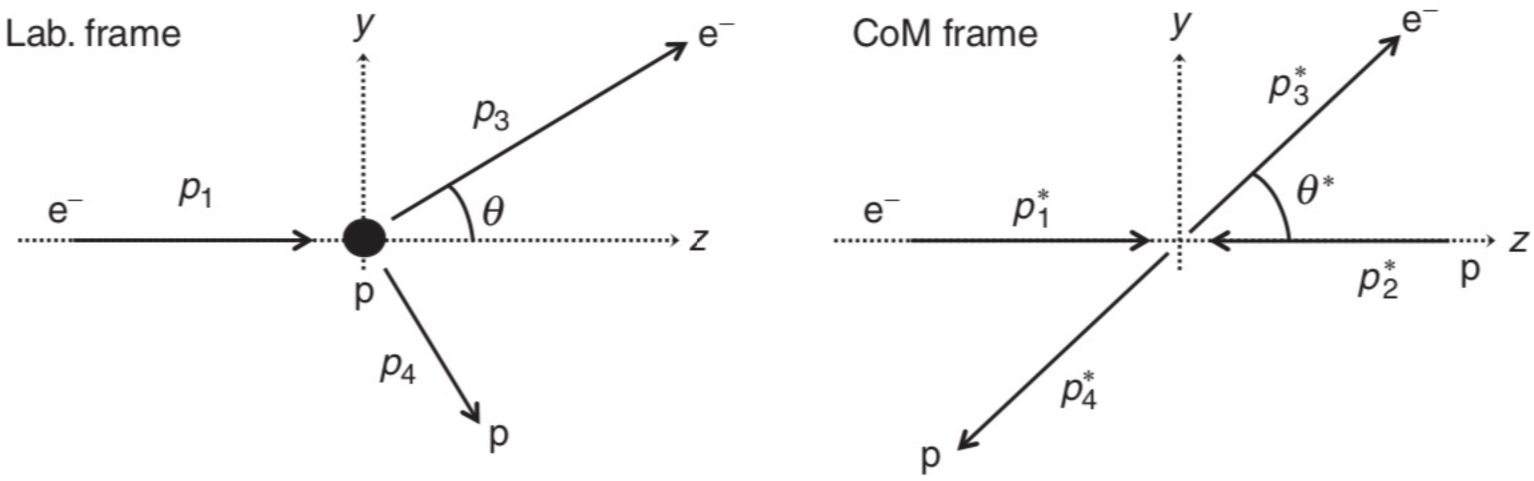
- Let's look at the lab frame now



We have then $$\mathrm{p}_i^{*2} = \frac{E_1^2 m_\mathrm{p}^2}{s}$$

Differential cross section

- Let's look at the lab frame now



We want to find the differential cross section in the lab frame:

$$\frac{d\sigma}{d\Omega} = \frac{d\sigma}{dt}\left|\frac{dt}{d\Omega}\right| = \frac{1}{2\pi}\frac{dt}{d(\cos\theta)}\frac{d\sigma}{dt}$$

Differential cross section

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$$\frac{d\sigma}{d\Omega} = \frac{d\sigma}{dt}\left|\frac{dt}{d\Omega}\right| = \frac{1}{2\pi}\frac{dt}{d(\cos\theta)}\frac{d\sigma}{dt}$$

$$t = (p_1 - p_3)^2 \approx -2E_1E_3(1-\cos\theta)$$

$$t = (p_2 - p_4)^2 = 2m_p^2 - 2p_2\cdot p_4 = 2m_p^2 - 2m_pE_4 = -2m_p(E_1 - E_3)$$

$$E_3 = \frac{E_1 m_p}{m_p + E_1 - E_1\cos\theta}$$

$$\begin{aligned}
p_1 &\approx (E_1, 0, 0, E_1),\\
p_2 &= (m_p, 0, 0, 0),\\
p_3 &\approx (E_3, 0, E_3\sin\theta, E_3\cos\theta),\\
p_4 &= (E_4, \mathbf{p}_4).
\end{aligned}$$

Differential cross section

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$$t = (p_1 - p_3)^2 \approx -2E_1E_3(1-\cos\theta)$$

$$\frac{dt}{d(\cos\theta)} = 2m_p\frac{dE_3}{d(\cos\theta)}$$

$$t = (p_2 - p_4)^2 = 2m_p^2 - 2p_2\cdot p_4 = 2m_p^2 - 2m_pE_4 = -2m_p(E_1 - E_3)$$

$$E_3 = \frac{E_1 m_p}{m_p + E_1 - E_1\cos\theta}$$

$$\frac{dE_3}{d(\cos\theta)} = \frac{E_1^2 m_p}{(m_p + E_1 - E_1\cos\theta)^2} = \frac{E_3^2}{m_p}.$$

$$\begin{aligned} p_1 &\approx (E_1, 0, 0, E_1),\\ p_2 &= (m_p, 0, 0, 0),\\ p_3 &\approx (E_3, 0, E_3\sin\theta, E_3\cos\theta),\\ p_4 &= (E_4, \mathbf{p}_4). \end{aligned}$$

Differential cross section

- We want to find the differential cross section in the lab frame:

$$\frac{d\sigma}{d\Omega} = \frac{d\sigma}{dt}\left|\frac{dt}{d\Omega}\right| = \frac{1}{2\pi}\frac{dt}{d(\cos\theta)}\frac{d\sigma}{dt}$$

$$t = (p_1 - p_3)^2 \approx -2E_1E_3(1-\cos\theta)$$

$$\frac{dt}{d(\cos\theta)} = 2m_p\frac{dE_3}{d(\cos\theta)}$$

$$t = (p_2 - p_4)^2 = 2m_p^2 - 2p_2\cdot p_4 = 2m_p^2 - 2m_pE_4 = -2m_p(E_1 - E_3)$$

$$\frac{dt}{d(\cos\theta)} = 2E_3^2$$

$$\begin{aligned} p_1 &\approx (E_1, 0, 0, E_1),\\ p_2 &= (m_p, 0, 0, 0),\\ p_3 &\approx (E_3, 0, E_3\sin\theta, E_3\cos\theta),\\ p_4 &= (E_4, \mathbf{p}_4). \end{aligned}$$

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$$\frac{d\sigma}{d\Omega} = \frac{1}{2\pi}2E_3^2\frac{d\sigma}{dt} = \frac{E_3^2}{64\pi^2 s\, p_i^{*2}}|\mathcal{M}_{fi}|^2$$

$$\frac{d\sigma}{d\Omega} = \frac{1}{64\pi^2}\left(\frac{E_3}{m_p E_1}\right)^2|\mathcal{M}_{fi}|^2$$

$$\frac{d\sigma}{d\Omega} = \frac{1}{64\pi^2}\left(\frac{1}{m_p + E_1 - E_1\cos\theta}\right)^2|\mathcal{M}_{fi}|^2$$

in terms of initial energy and scattering angle

Particle decays and scattering - summary

- Decay rate a-> 1+2:

$$\Gamma_{fi} = \frac{p^*}{32\pi^2 m_a^2} \int |\mathcal{M}_{fi}|^2 \, d\Omega$$

$$p^* = \frac{1}{2m_a}\sqrt{\left[(m_a^2 - (m_1 + m_2)^2\right]\left[m_a^2 - (m_1 - m_2)^2\right]}$$

- Differential cross section for a+b->c+d in the C.M. frame:

$$\frac{d\sigma}{d\Omega^*} = \frac{1}{64\pi^2 s}\frac{p_f^*}{p_i^*}|\mathcal{M}_{fi}|^2$$

- For ep elastic scattering in the Lab. Frame:

$$\frac{d\sigma}{d\Omega} = \frac{1}{64\pi^2}\left(\frac{E_3}{m_p E_1}\right)^2 |\mathcal{M}_{fi}|^2$$