II/ Aproximación de Campo Medio (mean field)

→ Para el modelo de Ising.

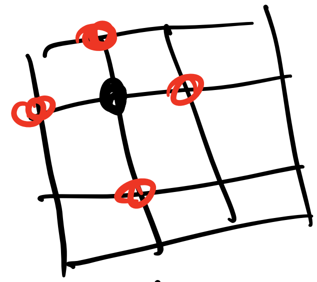
$$H = -J \sum_{\langle ij \rangle} \sigma_i \sigma_j - B \sum_i \sigma_i$$

en una red regular, con un número de coordinación z

z := # de vecinos de cada sitio

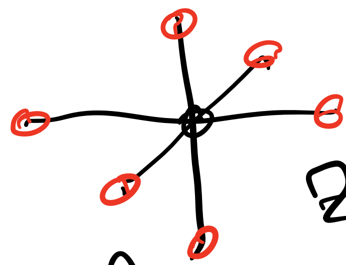
por ejemplo

- red cuadrada:



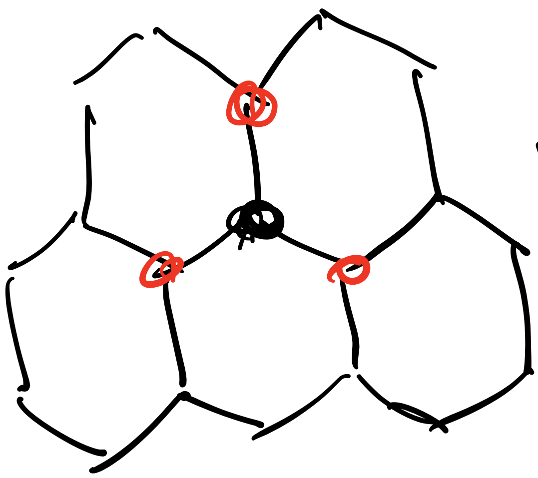
$z = 4$

- red cúbica



$z = 6$

- red hexagonal



$z = 3$

→ Aproximación: no tomar en cuenta las fluctuaciones de los σ_i:

$\forall \; \langle \sigma_i \rangle = m \quad , \quad \sigma_i = \pm 1$

N = # de espinos

$\forall i,\quad \boxed{\sigma_i = m + \underline{\delta\sigma_i}}$

{fluctuaciones alrededor del promedio (m)}

$$Z = \sum_{\{\sigma\}}' e^{-\beta H} = \sum_{\{\sigma\}}' e^{\frac{J}{k_B T}\sum_{\langle ij\rangle}\sigma_i\sigma_j + \frac{B}{k_B T}\sum_i \sigma_i} \qquad Z = e^{-\beta F}$$

$$= \sum_{\{\sigma\}}' e^{\underline{K\sum_{\langle ij\rangle}\sigma_i\sigma_j + h\sum_i \sigma_i}}$$

$$K = \frac{J}{k_B T}, \quad h = \frac{B}{k_B T}$$

$$K\sum_{\langle ij\rangle}'\sigma_i\sigma_j = K\sum_{\langle ij\rangle}'(m+\delta\sigma_i)(m+\delta\sigma_j)$$

$$= K\sum_{\langle ij\rangle}'\left[m^2 + m\,\delta\sigma_i + m\,\delta\sigma_j + \cancel{\delta\sigma_i\,\delta\sigma_j}\right]$$

Aproximación

$$\to K \sum_{\langle ij \rangle} \left[m^2 + m\,\delta\sigma_i + m\,\delta\sigma_j \right]$$

$$\delta\sigma_i = \sigma_i - m$$

$$= K \sum_{\langle ij \rangle}' \left[-m^2 + m\sigma_i + m\sigma_j \right]$$

$$\sum_{\langle ij \rangle}' m^2 = (\#\text{ enlaces})\, m^2 = \frac{Nz}{2} m^2$$

$$K \sum_{\langle ij \rangle} \sigma_i \sigma_j$$

$$\Rightarrow \simeq -K \frac{Nzm^2}{2} + Kmz \sum_i' \sigma_i$$

$$K \sum_{\langle ij \rangle} \sigma_i \sigma_j + h \sum_i \sigma_i$$

$$\simeq -\frac{Nzm^2K}{2} + h_{eff} \sum_i \sigma_i$$

donde $h_{eff} = h + KmZ$

$$\text{y } \boxed{Z = \sum_{\{\sigma\}} \underbrace{e^{-\frac{KNZm^2}{2}} e^{h_{eff} \sum_i \sigma_i}}_{Z_{CM}}}$$

$$Z_{CM} = e^{-\frac{KNZm^2}{2}} \sum_{\{\sigma\}} \prod_i e^{h_{eff}\sigma_i}$$

$$= e^{-\frac{KNZm^2}{2}} \prod_{i=1}^{N} \sum_{\sigma_i = \pm 1} e^{h_{eff}\sigma_i}$$

$$= \prod_{i=1}^{N} (Z_i)$$

$$\text{donde } Z_i = e^{-\frac{KZm^2}{2}} \sum_{\sigma_i = \pm 1} e^{h_{eff}\sigma_i}$$

$$Z_{CM} = (Z_1)^N \text{ con } Z_1 = e^{-\frac{KZm^2}{2}} \sum_{\sigma_1 = \pm 1} e^{h_{eff}\sigma_1}$$

$$\Rightarrow \boxed{Z_1 = e^{-\frac{z m^2 K}{2}}\, 2\, ch(h_{eff})}$$

$$\langle \sigma_1 \rangle = m = \frac{1}{Z_1} \sum_{\sigma_1 = \pm 1} e^{-\frac{z m^2 K}{2}} e^{h_{eff} \sigma_1} \sigma_1$$

$$= \frac{1}{Z_1} e^{-\frac{z m^2 K}{2}}\, 2\, sh(h_{eff})$$

$$\Rightarrow \boxed{m = tgh(h_{eff})}$$

$$\Rightarrow \boxed{m = tgh(h + Kmz)}$$

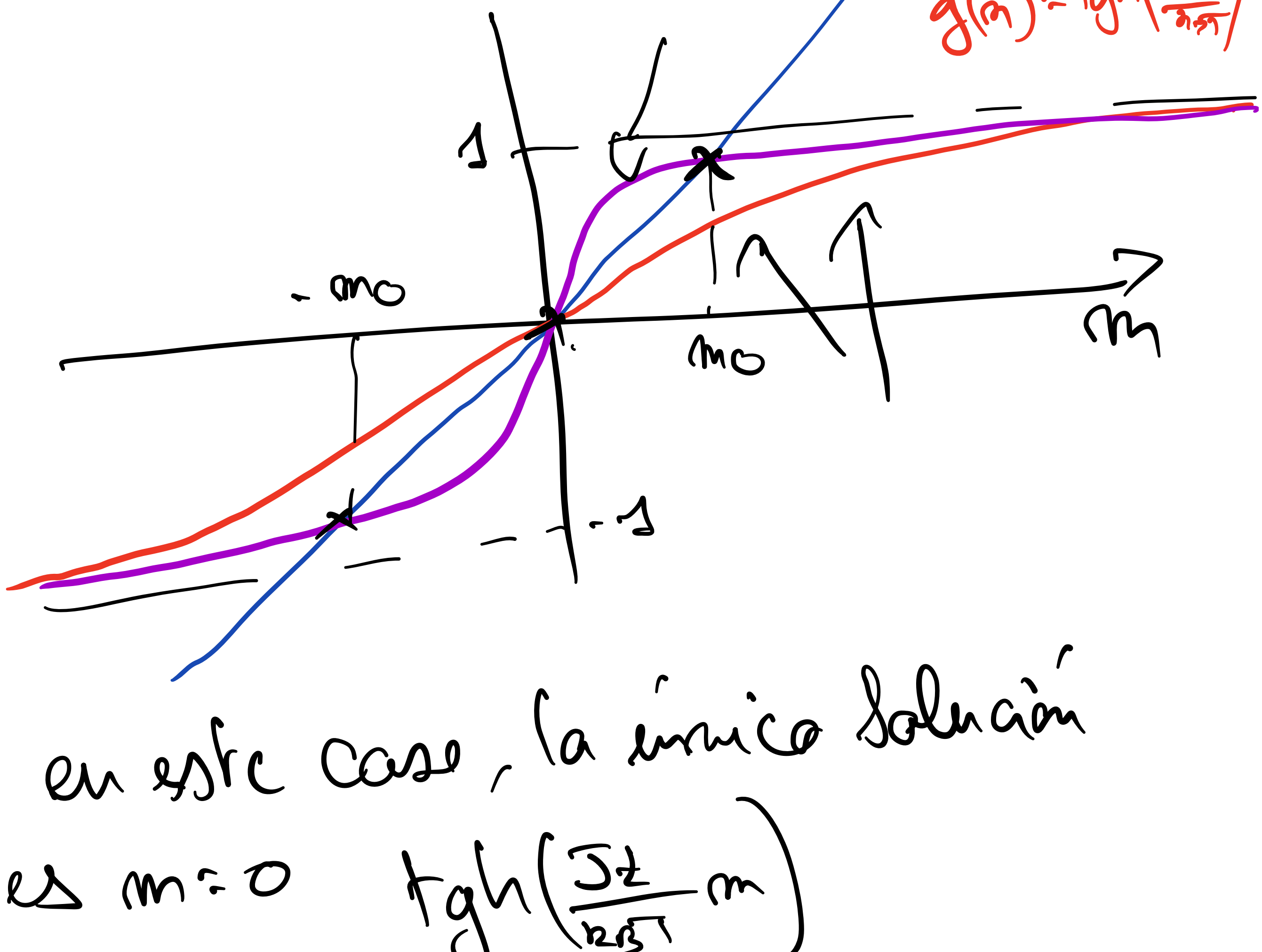
ecuación de autoconsistencia de la aproximación.

¿hay transición de fase?

→ hay que poner $h = 0$ $(B = 0)$

$\Rightarrow$ $m = \operatorname{tgh}(K m z) = \operatorname{tgh}\left(\frac{Jz}{k_B T} m\right)$

resolución gráfica:



en este caso, la única solución es $m = 0$

$\operatorname{tgh}\left(\frac{Jz}{k_B T} m\right)$

pendiente en $m=0$ $= \frac{Jz}{k_B T}$

Tres soluciones posibles

$m = 0, \pm m_0$

→ las soluciones termodinámicas

- te estables son $m = \pm m_0$

→ ruptura espontánea de $\mathbb{Z}_2$

→ Fase FERRO

O sea que para $\left\{ \boxed{\begin{array}{l} T > T_c \Rightarrow m = 0 \\ T < T_c \Rightarrow m = \pm m_0 \end{array}} \right.$

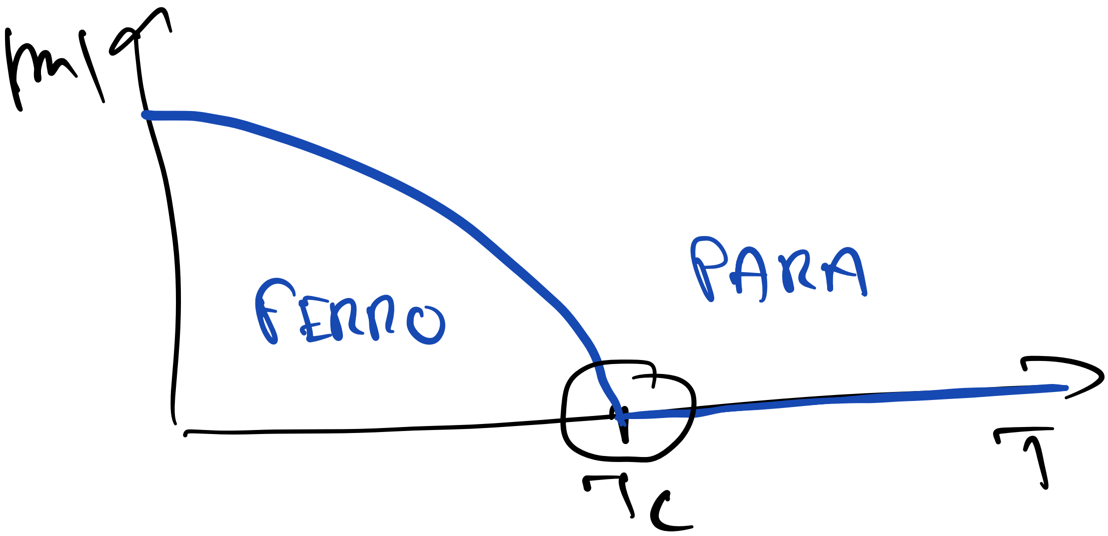
¿T_c?

T_c es tal que la pendiente

$\operatorname{tgh}\left(\frac{J z m}{k_B T}\right)$ para $m = 0$ es 1

la pendiente (en $m = 0$) es $\frac{J z}{k_B T}$

$$\Rightarrow \boxed{T_c = \frac{Jz}{k_B}}$$

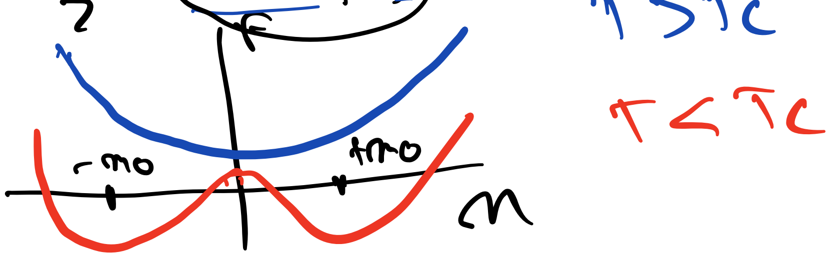


$$F = N\left[-k_B T \ln 2 + \frac{Jz\,m^2}{2} - k_B T \ln\left[\mathrm{ch}\left(\frac{Jzm}{k_B T}\right)\right]\right]$$

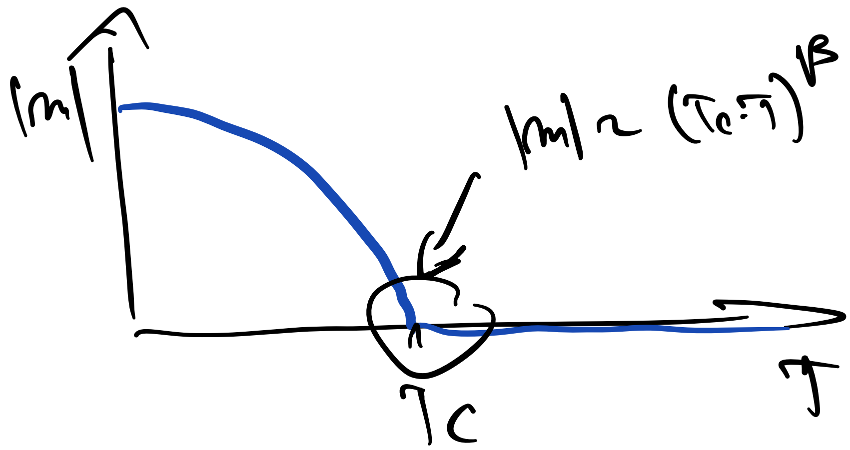
si $m \simeq 0$

$$F \simeq \text{cte} + \frac{Jz}{2}\left(1 - \frac{zJ}{k_B T}\right) m^2 + O(m^4)$$

$$= \text{cte} + \frac{Jz}{2}\left(1 - \frac{T_c}{T}\right) m^2 + O(m^4)$$



exponente β:



$$m = \tanh\left(\frac{Jz}{k_B T} m\right) \qquad m \simeq 0$$

$$\Rightarrow \cancel{m} \simeq \frac{Jz}{k_B T} \cancel{m} - \frac{\left(\frac{Jz}{k_B T}\right)^3 m^{\cancel{3}\,2}}{3} + O(m^5)$$

$$m^2 \frac{\left(\frac{Jz}{k_B T}\right)^3}{3} = \frac{Jz}{k_B T} - 1 = \left(\frac{T_c}{T} - 1\right)$$

$$\Rightarrow m \sim (T_c - T)^\beta \text{ con } \boxed{\beta = \frac{1}{2}}$$

¿Qué tan fiable es esta aproximación?

→ depende de la dimensionalidad

* $D = 1$ 

$z = 2$

→ desastre!

→ No hay fase Ferro!

* $D = 2$

Sí hay fase Ferro, pero T_c y β son incorrectos

* $D = 3$

Idem que para $D = 2$

* $D \geq 4$
→ si hay Mean Field

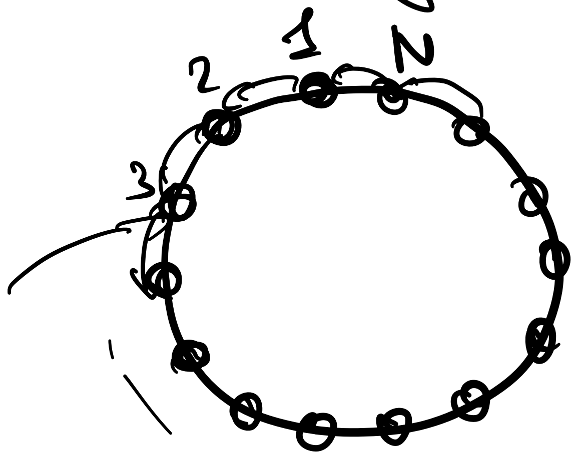
T_c no es correcto, pero los exp. críticos β, η, γ etc...
son correctos!

$\boxed{\begin{array}{l}\text{* Un modelo donde con acoplos de alcance infinito:} \\ \to \text{Mean-field} \to \text{exacto} \\ T_c \checkmark, \beta, \eta, \gamma \checkmark \text{ etc.}\end{array}}$

XII Solución exacta en 1-D

1) Método de la matriz de Transferencia

Cadena de Ising, C.B.P.



$\sigma_1, \sigma_2, \ldots, \sigma_N$

C.B.P. $\sigma_{N+1} = \sigma_1$

$$H = -J \sum_{i=1}^{N} \sigma_i \sigma_{\underset{N+1}{i+1}} - B \sum_{i=1}^{N} \sigma_i$$

$$K = \frac{J}{k_B T} \qquad h = \frac{B}{k_B T}$$

$$Z = \sum_{\{\sigma\}} e^{K \sum_{i=1}^{N} \sigma_i \sigma_{i+1} + h \sum_{i=1}^{N} \sigma_i}$$

$$= \sum_{\{\sigma\}} \prod_{i=1}^{N} e^{K\sigma_i\sigma_{i+1} + \frac{h}{2}\sigma_i + \frac{h}{2}\sigma_{i+1}}$$

$\underline{\text{def}}$ La matriz de transferencia (2×2)

$$T(\sigma_i, \sigma_{i+1}) =$$

$= \pm 1$

$\sigma_i = \sigma_{i+1} = 1$

$\sigma_i = 1$
$\sigma_i = -1$

$\sigma_i = -1$
$\sigma_{i+1} = 1$

$\sigma_i = \sigma_{i+1} = -1$

$$T = \begin{pmatrix} e^{k+h} & e^{-k} \\ e^{-k} & e^{k-h} \end{pmatrix}$$

$$Z = \sum_{\substack{i_1 = 1,2 \\ i_2 = 1,2 \\ \vdots \\ i_N = 1,2}} T_{i_1, i_2} T_{i_2, i_3} T_{i_3, i_4} \cdots T_{i_N, i_N}$$

$$T_{i_N, i_1}$$

$$\boxed{Z = \mathrm{tr}\{T^N\}}$$

si ϵ_1 y ϵ_2 son los autovalores de T

$$\epsilon_{1/2} = e^{k} ch(h) \pm \left[e^{2k} sh^2(h) + e^{-2k}\right]^{1/2}$$

$$\boxed{Z = \epsilon_1^N + \epsilon_2^N} \qquad \epsilon_1 > \epsilon_2$$

si $N \gg 1$

$$Z = \epsilon_1^N \left(1 + \left(\frac{\epsilon_2}{\epsilon_1}\right)^N\right) \xrightarrow[N \to \infty]{} \epsilon_1^N$$

$$F = -k_B T \ln Z = -k_B T \ln \epsilon_1$$

$$y \ \langle \sigma_i \rangle = m = \frac{1}{N} \frac{\partial}{\partial h} \ln Z = \frac{\partial}{\partial h} \ln \epsilon_1$$

$$\boxed{m = \frac{\operatorname{sh}(h)}{k_B T \sqrt{\operatorname{sh}^2(h) + e^{-4K}}}}$$

obs si tomamos $B = 0$ $(h = 0)$

$\to \boxed{m = 0 \ \forall T}$!

no hay fase ferro en 1-D

2) funciones de correlación

$$\langle \sigma_i \sigma_j \rangle = \frac{1}{Z} \sum_{\{\sigma\}} e^{-\beta H} \, \sigma_i \sigma_j \qquad j > i$$

para simplificar $B = 0$ $(h = 0)$

$$T = \begin{pmatrix} e^{K} & e^{-K} \\ e^{-K} & e^{K} \end{pmatrix}$$

$$\Sigma = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$\langle \sigma_i \sigma_j \rangle = \frac{1}{Z} \sum_{\substack{l_1 = 1,2 \\ l_2 \\ \vdots}} T_{l_1, l_2} T_{l_2, l_3} \cdots T_{i-1, i} \Sigma \cdots T_{i, i+1} \cdots T_{j-1, j} \Sigma \, T_{j, j+1} \cdots T_{l_N, 1}$$

$$= \frac{1}{Z} \operatorname{tr} \left(T^{i} \Sigma T^{j-i} \Sigma T^{N-j} \right)$$

Ne diagonalija T

$$\Theta = \begin{pmatrix} \frac{1}{\sqrt2} & \frac{1}{\sqrt2} \\ \frac{1}{\sqrt2} & -\frac{1}{\sqrt2} \end{pmatrix}, \quad \Theta\Theta = \mathbb{1}$$

$$\Theta T \Theta = \begin{pmatrix} \epsilon_1 & 0 \\ 0 & \epsilon_2 \end{pmatrix} = \tilde{T} \text{ cn}$$

$$\epsilon_1 = 2\,\mathrm{Ch}\,K \;;\; \epsilon_2 = 2\,\mathrm{Sh}(K)$$

$$\Theta \Sigma \Theta = \hat{\Sigma} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$\mathrm{Tr}\{T^\alpha \Sigma T^\beta\} = \mathrm{Tr}\{\Theta T \underbrace{\Theta\Theta}_{\mathbb{1}} T \Theta\Theta \ldots$$

$$= \mathrm{Tr}\{\hat{T}^\alpha \hat{\Sigma} \hat{T}^\beta\}$$

$$D\langle \sigma_i \sigma_j \rangle = \frac{1}{Z} \mathrm{Tr}\{\hat{T}^i \hat{\Sigma} \hat{T}^{j-i} \hat{\Sigma} \hat{T}^{N-j}\}$$

$$= \frac{1}{2}\, \mathrm{tr}\left\{ \begin{pmatrix} \epsilon_1^{i} & 0 \\ 0 & \epsilon_2^{i} \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \epsilon_1^{j-i} & 0 \\ 0 & \epsilon_2^{j-i} \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \epsilon_1^{N-j} & 0 \\ 0 & \epsilon_2^{N-j} \end{pmatrix} \right.$$

$$= \langle \sigma_i \sigma_j \rangle = \frac{\epsilon_1^{N-j+i}\, \epsilon_2^{j-i} + \epsilon_2^{N-j+i}\, \epsilon_1^{j-i}}{\epsilon_1^{N} + \epsilon_2^{N}}$$

si $N \to \infty$

$$\langle \sigma_i \sigma_j \rangle \xrightarrow[N\to\infty]{} \frac{\cancel{\epsilon_1^{N}} \left[\left(\frac{\epsilon_2}{\epsilon_1}\right)^{j-i} + \left(\frac{\epsilon_2}{\epsilon_1}\right)^{N} \left(\frac{\epsilon_1}{\epsilon_2}\right)^{j-i} \right]}{\cancel{\epsilon_1^{N}} \left(1 + \left(\frac{\epsilon_2}{\epsilon_1}\right)^{N}\right)}$$

$\left(\frac{\epsilon_2}{\epsilon_1}\right)^{N} \to 0$

$N \to \infty$

$$\langle \sigma_i \sigma_j \rangle \longrightarrow \left(\frac{\epsilon_2}{\epsilon_1}\right)^{j-i} = \left[\tanh(K)\right]^{j-i}$$

$$= e^{-\frac{d_{j-i}}{\xi}}$$

$$\xi = \frac{1}{\ln(\coth(K))} = \frac{1}{\ln\left(\coth\left(\frac{J}{k_B T}\right)\right)}$$

$\xi \to 0$ si $T \to \infty$

$\xi \to \infty$ si $T \to 0$