IV) Del modelo de Ising a una Teoría de Campos

1) La Transformación de Hubbard-Stratonovich

$$H = -\frac{1}{2} \sum_{\forall i,j} J_{ij}\, \sigma_i \sigma_j \qquad J_{ij} = J_{ji} \in \mathbb{R}$$

$$Z = \sum_{\{\sigma\}} e^{\frac{\beta}{2} \sum_{i,j} \sigma_i J_{ij} \sigma_j}$$

$$\sigma_i \underset{i=1,\dots,N}{} = \begin{pmatrix} \sigma_1 \\ \sigma_2 \\ \vdots \\ \sigma_N \end{pmatrix} = \sigma \qquad \sigma^T = (\sigma_1, \sigma_2, \dots, \sigma_N)$$

J matriz $N \times N$ con entradas J_{ij}

$$\sum_{i,j} \sigma_i J_{ij} \sigma_j = \sigma^T J \sigma$$

$$= (\sigma_1, \sigma_2, \ldots, \sigma_N) \begin{pmatrix} J_{11} & J_{12} & \cdots \\ J_{21} & & \\ \vdots & & \end{pmatrix}_{N \times N} \begin{pmatrix} \sigma_1 \\ \vdots \\ \sigma_N \end{pmatrix}$$

Sin pérdida de generalidad, todos los autovalores de J son positivos.

$$J \to \underbrace{J + c\,\mathbb{1}_{N \times N}}_{c > 0} = \tilde{J}$$

$$H \to \tilde{H} = H + \underbrace{\sum_i c\, \overset{1}{\overset{\shortparallel}{\sigma_i^2}}}$$

$$\tilde{H} = H + Nc$$

$$\boxed{J > 0} \Rightarrow \exists\, J^{-1} > 0$$

<u>obs</u> $(J_{ij}) = J$ es una matriz <u>real</u> <u>y simétrica</u>

<u>ej</u>:

$$I = \int_{-\infty}^{\infty} \prod_{i=1}^{N} dx_i \; e^{-\frac{1}{2}\sum_{i,j} x_i A_{ij} x_j + \sum_i x_i y_i}$$

con A simétrica y > 0. $A^{-1}A = \mathbb{1}_{N\times N}$

$y_1, \dots, y_N$ reales

$$I = \sqrt{\frac{(2\pi)^N}{\mathrm{Det}(A)}}\; e^{\frac{1}{2}\sum_{i,j} y_i (A^{-1})_{ij} y_j}$$

Indicaciones:

* hacer un cambio de variables en las var. x_i:

$$\begin{pmatrix} x_1 \\ \vdots \\ x_N \end{pmatrix} = \vec{x} = \vec{\tilde{x}} + A^{-1}\vec{y}$$

$$\vec{y} = \begin{pmatrix} y_1 \\ \vdots \\ y_N \end{pmatrix}$$

$$\Rightarrow I = e^{\frac{1}{2} y^T A^{-1} y} \int_{-\infty}^{\infty} \prod_i d\tilde{x}_i \, e^{-\frac{1}{2} \tilde{x}^T A \tilde{x}}$$

* $\exists\ \Theta$ tq $\Theta^T \Theta = \mathbb{1}_{N\times N}$

y $\Theta^T A \Theta = D = \begin{pmatrix} d_1 & & 0 \\ & d_2 & \\ 0 & & \ddots \; d_N \end{pmatrix}$

$$\Rightarrow \tilde{x} = \Theta \tilde{\tilde{x}}$$

$\llcorner \det \Theta = 1$

$$* \quad \int_{-\infty}^{\infty} dx \, e^{-\frac{\lambda x^2}{2}} = \sqrt{\frac{2\pi}{\lambda}}$$

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con la identificación

$$\beta J \longleftrightarrow A^{-1}$$

$$\sigma_i \longleftrightarrow y_i$$

$$\phi_i \longleftrightarrow x_i$$

$$\boxed{e^{\frac{\beta}{2}\sigma^T J \sigma} = (2\pi)^{-N/2} \left[\mathrm{Det}(\beta J)\right]^{-1/2} \times \int_{-\infty}^{\infty} \prod_{i=1}^{N} d\phi_i \, e^{-\frac{1}{2\beta}\phi^T J^{-1} \phi} \, e^{\sum_i \sigma_i \phi_i}}$$

$\sigma_i = \pm 1$

$\phi_i \in]-\infty, \infty[$

$$Z = (2\pi)^{-\frac{N}{2}} \left[\mathrm{Det}(\beta J)\right]^{-\frac{1}{2}}$$
$$\times \int_{-\infty}^{\infty} \prod_{i=1}^{N} d\phi_i \; e^{-\frac{1}{2\beta} \sum_{i,j} \phi_i (J^{-1})_{ij} \phi_j} \sum_{\{\sigma\}} e^{\sum_{i=1}^{N} \sigma_i \phi_i}$$

Obs

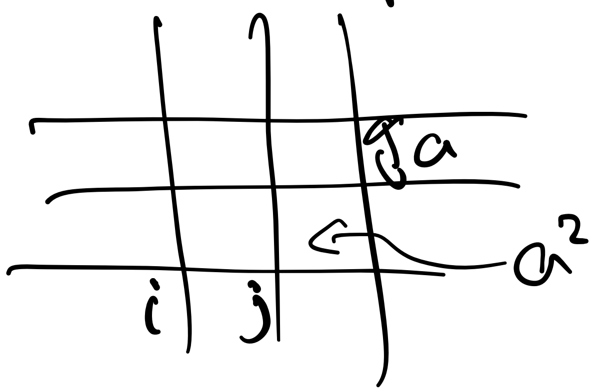
$$\sum_{\{\sigma\}} e^{\sum_{i=1}^{N} \sigma_i \phi_i} = \sum_{\{\sigma\}} \prod_{i=1}^{N} e^{\sigma_i \phi_i}$$
$$= \prod_i \left[\sum_{\sigma_i = \pm 1} e^{\sigma_i \phi_i}\right] = 2^N \prod_{i=1}^{N} \mathrm{ch}(\phi_i)$$

$$Z = (2\pi)^{-N/2} \left[\mathrm{Det}(\beta J)\right]^{-\frac{1}{2}}$$
$$\times \int \prod_i d\phi_i \; e^{-\frac{1}{2\beta} \phi^T J^{-1} \phi - V(\phi)}$$
$$V(\phi) = -N \ln 2 - \sum_{i=1}^{N} \ln\left[\mathrm{ch}(\phi_i)\right]$$

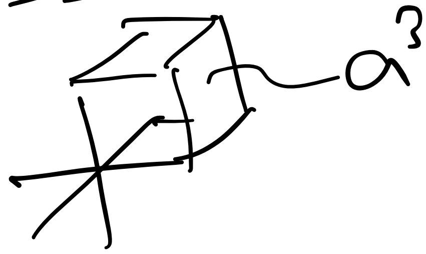
2] Límite al continuo   dimensión = d

Cuando $T \to T_c$, $\xi \to \infty$

si a es espaciamiento de la red



$\xi \gg a$



$i, j \to \vec{r}_i, \vec{r}_j$

$$\sum_i \to \frac{1}{a^d}\int d\vec{r}$$

$$\phi_i \to a^d \phi(\vec{r})$$

$$\delta_{i,j} \to a^d \delta(\vec{r}_i - \vec{r}_j)$$

$$\sum_j \delta_{i,j} = 1 \to \frac{1}{\cancel{a^d}}\int d\vec{r}_j \, \cancel{a^d}\, \delta(\vec{r}_i - \vec{r}_j) = 1$$

$J_{ij} \Rightarrow J(|\vec{r}_i - \vec{r}_j|)$

invariancia por traslación y rotación.

$J^{-1}_{ij} \Rightarrow J^{-1}(|\vec{r}_i - \vec{r}_j|)$

$$\sum_l J_{il} J^{-1}_{lj} = \delta_{ij}$$

$$\Rightarrow \frac{1}{a^d} \int d\vec{r}\, J(|\vec{r} - \vec{r}_i|) J^{-1}(|\vec{r} - \vec{r}_j|)$$
$$= a^d \delta(\vec{r}_i - \vec{r}_j) \quad (*)$$

→ hacemos la T.F.

$$\hat{\phi}(\vec{k}) = \int d\vec{r}\; e^{i\vec{k}\cdot\vec{r}} \phi(\vec{r})$$

$$\phi(\vec{r}) = \frac{1}{(2\pi)^d} \int d\vec{k}\, e^{-i\vec{k}\cdot\vec{r}} \hat{\phi}(\vec{k})$$

obs como $\phi(\vec{r}) \in \mathbb{R} \Rightarrow \hat{\phi}(-\vec{k}) = \hat{\phi}^*(\vec{k})$

$$\boxed{\hat{J}(\vec{k}) = \int d\vec{r}\, e^{i\vec{k}\cdot\vec{r}}\, J(|\vec{r}|)}$$

$x_1 \to x_1$

$x' \leftarrow$

$$J(\vec{r}) = \frac{1}{(2\pi)^d}\int d\vec{k}\, e^{-i\vec{k}\cdot\vec{r}}\, \hat{J}(\vec{k})$$

$$\hat{J}^{-1}(\vec{k}) = \int d\vec{r}\, e^{i\vec{k}\cdot\vec{r}}\, J^{-1}(\vec{r})$$

ej:

$$\int d\vec{r}\, e^{i(\vec{k}-\vec{k}')\cdot\vec{r}} = (2\pi)^d\, \delta(\vec{k}-\vec{k}')$$

$$\int d\vec{k}\, e^{i\vec{k}\cdot(\vec{r}_i-\vec{r}_j)} = (2\pi)^d\, \delta(\vec{r}_i-\vec{r}_j)$$

$$\hat{J}^{-1}(\vec{k}) = \frac{a^{2d}}{\hat{J}(\vec{k})} \Leftarrow$$

$$\sum_{i,j} \phi_i J^{-1}_{ij} \phi_j \to$$

$$\int d\vec{r}\, d\vec{r}'\, \phi(\vec{r})\, J^{-1}(|\vec{r}-\vec{r}'|)\, \phi(\vec{r}')$$

$$= \frac{1}{(2\pi)^d} \int d\vec{k}\, |\hat{\phi}(\vec{k})|^2\, \hat{J}^{-1}(\vec{k})$$

$$= \frac{a^{2d}}{(2\pi)^d} \int d\vec{k}\, \frac{|\hat{\phi}(\vec{k})|^2}{\hat{J}(\vec{k})}$$

→ el comportamiento a grandes distancias, k pequeño.

$$\hat{J}(\vec{k}) = \hat{J}(\vec{0}) + \sum_{\alpha=1,\dots,d} \cancelto{0}{k^\alpha \frac{\partial \hat{J}}{\partial k^\alpha}\Big|_{k=0}}$$

$$+ \frac{1}{2} \sum_{\alpha,\beta} k^\alpha k^\beta \underbrace{\frac{\partial^2 \hat{J}}{\partial k^\alpha \partial k^\beta}}_{\hookrightarrow\, 0 \text{ si } \alpha \neq \beta} + \dots$$

$$\hat{J}(\vec{k}) = \hat{J}(\vec{0}) + \frac{1}{2}\vec{k}^2\left(\frac{\partial^2 \hat{J}}{\partial k_1^2}\right) + \dots$$

$$\left(\frac{\partial^2 \hat{J}}{\partial k_1^2}\right) = -\int d\vec{r}\, x_1^2 J(|\vec{r}|) = -\int d\vec{r}\, x_2^2 J(|\vec{r}|) = \dots = -\frac{1}{d}\int d\vec{r}\,(\vec{r}^2) J(|\vec{r}|)$$

def : alcance efectivo de $J(\vec{r})$: ℓ

$$\ell^2 = \frac{\frac{1}{2d}\int d\vec{r}\; \vec{r}^2 J(|\vec{r}|)}{\underbrace{\int d\vec{r}\; J(\vec{r})}_{\hookrightarrow \hat{J}(\vec{0})}} > 0$$

$$\hat{J}(\vec{k}) = \hat{J}(\vec{0})\left[1 - \ell^2 \vec{k}^2 + \dots\right]$$

$$\frac{1}{[1 - \ell \vec{k}^2 + \dots]} \simeq \left[1 + \ell \vec{k}^2 + \dots\right]$$

$$\frac{1}{2\beta}\phi^T \mathcal{J}^{-1}\phi$$

$$\Rightarrow \frac{a^{2d}}{2\beta\hat{\mathcal{J}}(0)}\int\frac{d\vec{k}}{(2\pi)^d}\,|\hat{\phi}(\vec{k})|^2\left(1+\ell^2\vec{k}^2+\dots\right)$$

$$= \frac{a^{2d}}{2\beta\hat{\mathcal{J}}(0)}\int d\vec{r}\,\left[\phi^2(\vec{r})+\ell^2(\vec{\nabla}\phi)^2+\dots\right]$$

$$\underline{\ln\operatorname{ch}\phi} = \ln\left(1+\frac{\phi^2}{2}+\frac{\phi^4}{4!}+\dots\right)$$
$$= \frac{\phi^2}{2}-\frac{\phi^4}{12}+\dots$$

$$e^{-\frac{1}{2\beta}\phi^T\mathcal{J}^{-1}\phi - V(\phi)}$$

$$= e^{-\frac{a^{2d}}{2\beta\hat{\mathcal{J}}(0)}\int d\vec{r}\,\left[\phi^2+\ell^2(\vec{\nabla}\phi)^2\right]} \times e^{-\int d\vec{r}\,\left[\frac{a^d}{2}\phi^2 - \frac{a^{3d}}{12}\phi^4+\dots\right]}$$

$$\int_{-\infty}^{\infty} \prod_i d\phi_i \longrightarrow \int \mathcal{D}[\phi]$$

$$Z = \text{cte} \int \mathcal{D}\{\phi\}\, e^{-S\{\phi\}}$$

$d \qquad m^2 \qquad d \qquad d$

$$S\{\phi\} = \int d\vec{r} \left[\frac{c}{2} (\vec{\nabla}\phi)^2 + \frac{a_2}{2}\phi^2 + \frac{a^4}{4}\phi^4 \dots \right]$$

$$c = \frac{\ell^2}{\beta \tilde{J}(0)} \; ; \quad a_2 = \left(\frac{k_B T}{\tilde{J}(0)} - \frac{1}{a^d} \right) a^{2d}$$

$$a_4 = \frac{a^{3d}}{3} > 0$$

acción de Ginzburg-Landau

$$S\{\phi\} = \int d\vec{r} \left[\frac{c}{2} (\vec{\nabla}\phi)^2 + \frac{a_2}{2}\phi^2 + \frac{a_4}{4}\phi^4 \dots \right]$$

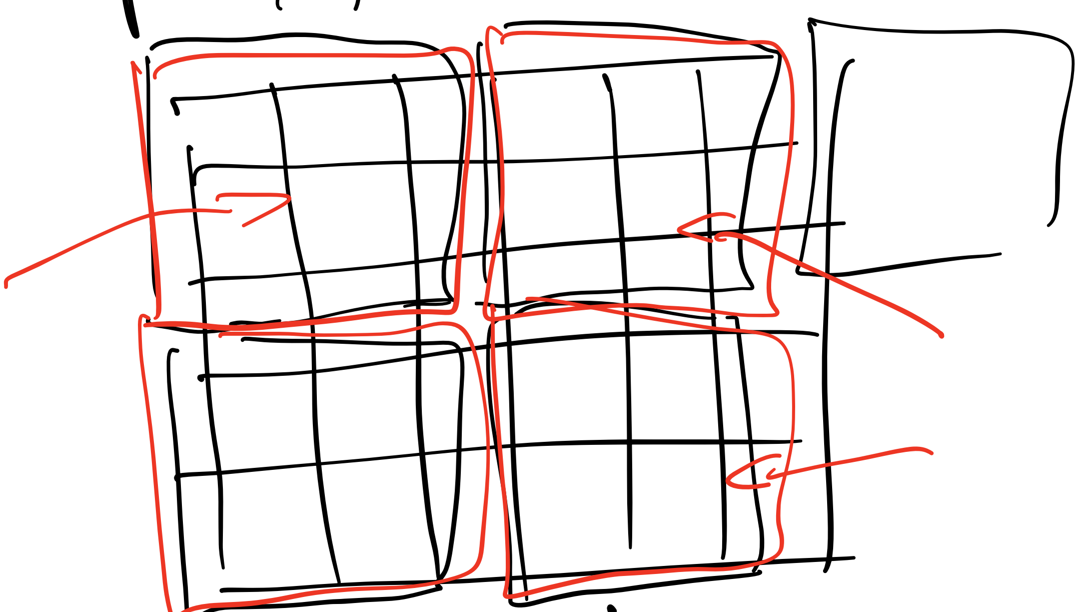
$$Z = \sum_{\{\sigma\}} e^{-\beta H}$$

$$\hookrightarrow Z = \int \mathcal{D}\{\phi\}\, e^{-S}$$

$$\sigma_i \pm 1 \longleftrightarrow \phi(\vec{r}) \in]-\infty, \infty[$$

$$\sum_{\{\sigma\}} \longleftrightarrow \int \mathcal{D}\{\phi\}$$

$$\beta H\{\sigma_i\} \longrightarrow S\{\phi\}$$



coarse-graining.

$\phi(\vec{r}) \rightarrow$ promedio de varios $\sigma_i = \pm 1$

<u>obs</u> $S[\phi]$ simetría $\mathbb{Z}_2$

$$\begin{cases} \phi(\vec{r}) \rightarrow \phi(\vec{r}) \\ \phi(\vec{r}) \rightarrow -\phi(\vec{r}) \end{cases}$$

$$a_2 = a(T - T_c)$$

$T > T_c \quad a_2 > 0$

$T < T_c \quad a_2 < 0$

El mínimo de $S[\phi]$

$$S[\phi] = \int d\vec{r} \left[c(\vec{\nabla}\phi)^2 + \frac{a_2}{2}\phi^2 + \frac{a_4}{4}\phi^4 \right]$$

$c, a_4 > 0$

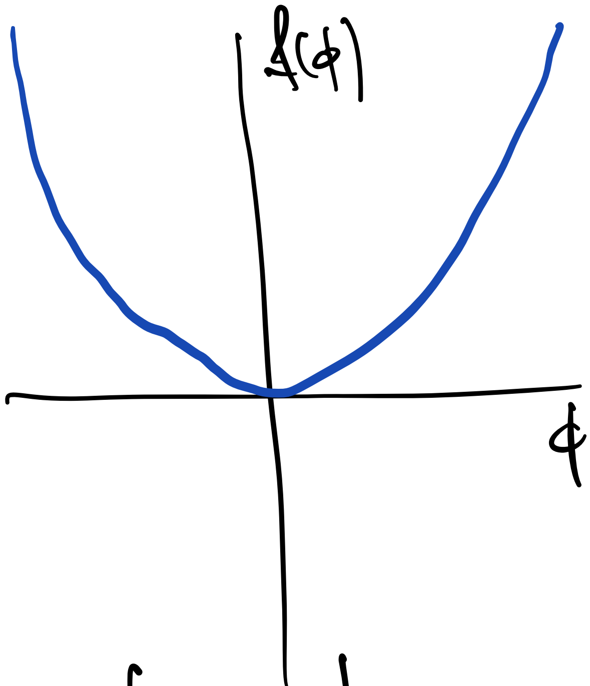
$c>0$ el mínimo

$\vec{\nabla}\phi = \vec{0}$! $\Rightarrow \phi$ es homogéneo.

$$\frac{a_2}{2}\phi^2 + \frac{a_4}{4}\phi^4$$

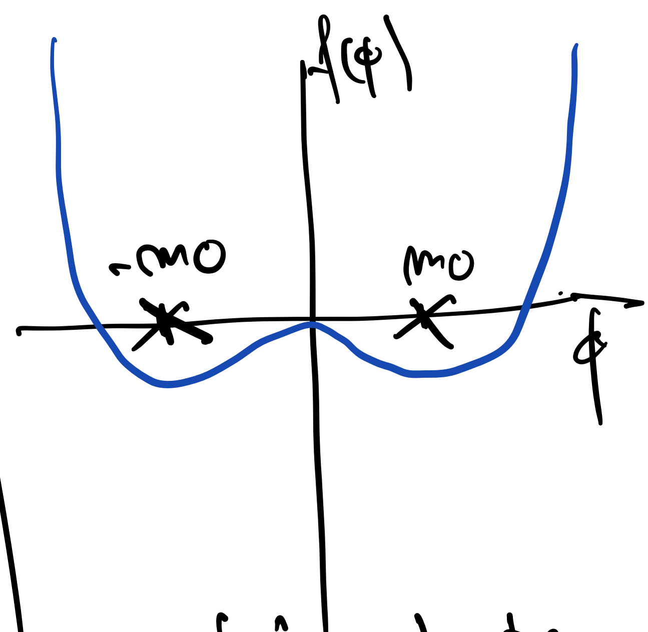
$$f(\phi) = \frac{a_2}{2}\phi^2 + \frac{a_4}{4}\phi^4 \quad \begin{cases} a_4 > 0 \\ a_2 > 0 \text{ si } T > T_c \\ a_2 < 0 \text{ si } T < T_c \end{cases}$$

$a_2 > 0 \ (T > T_c)$



mínimo $\phi = 0$

$a_2 < 0 \ (T < T_c)$



mínimo $\phi = \pm m_0$

$T > T_c \Rightarrow$ PARA

$T < T_c \Rightarrow$ FERRO