IV El Modelo Gaussiano

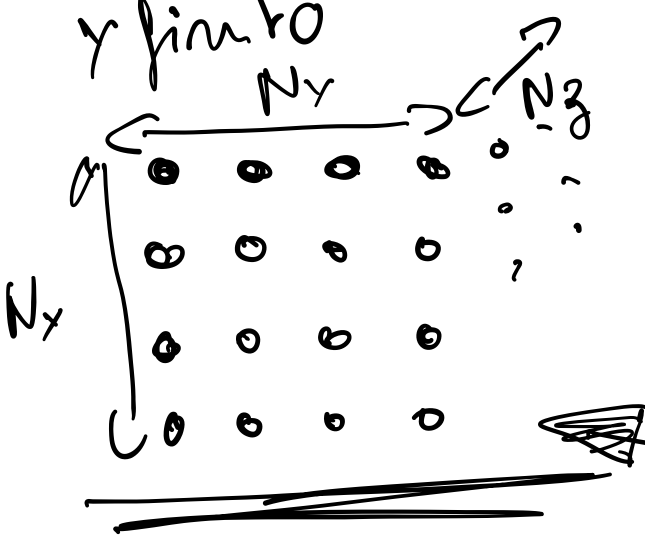
$$Z = \int \mathcal{D}[\phi]\, e^{-S[\phi]}$$

?

si $S[\phi]$ es cuadrática...

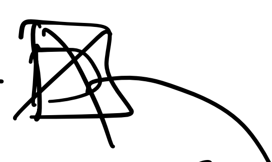
Integrales múltiples en $\mathbb{R}^n$ $\longrightarrow$ Integrales funcionales

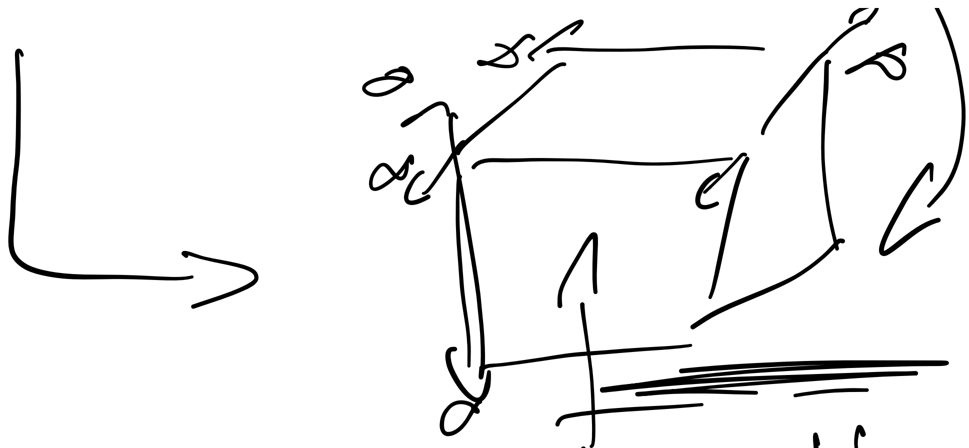
Sistema discreto y finito



N sitios

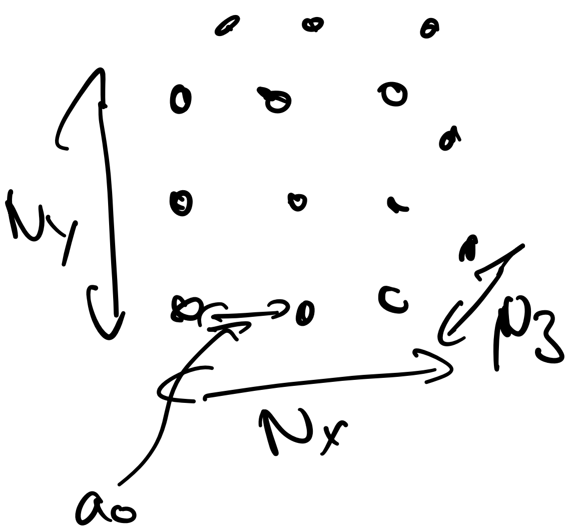
$N = N_x N_y N_z$ finito





sistema cartesiano.

1) Sistema discreto y finito



d dimensiones

$N = N_x N_y N_z$ etc..

red hipercúbica.

$$L_x = a_0 N_x$$
$$L_y = a_0 N_y$$
$$\vdots$$

$$\underline{V} = L_x L_y L_z \ldots = a_0^d \underline{N}$$

$$\vec r = m_x a_0 \hat e_x + m_y a_0 \hat e_y + m_z a_0 \hat e_z + \dots$$

$m_x = 0, 1, \dots, N_x - 1$, idem para m_y etc.

Condiciones de borde periódicas

$$\vec r \equiv \vec r + N_x a_0 \hat e_x$$
$$\equiv \vec r + N_y a_0 \hat e_y$$
etc...

en cada sitio, definimos

$\phi_{\vec r} \in \mathbb{R}$ $\quad \phi_{\vec r} = \phi_{\vec r + N_x a_0 \hat e_x}$

Modos de Fourier:

$$e^{i\vec k \cdot \vec r} \; ; \; \vec k = \frac{2\pi n_x}{L_x}\hat e_x + \frac{2\pi n_y}{L_y}\hat e_y + \dots$$

$n_x, n_y, \dots \in \mathbb{Z}$

$$e^{i\vec k\cdot(\vec r + L_x \hat e_x)} = e^{i\vec k\cdot\vec r}$$

$-\frac{N_x}{2} < n_x \leq \frac{N_x}{2}$ idem para n_y y n_z.

Primera zona de Brillouin

→ ortogonalidad de los modos de Fourier:

(Todo se muestra con $\forall \alpha \neq 1$

$$\sum_{n=0}^{N-1} \alpha^n = \frac{1-\alpha^N}{1-\alpha}$$

Se muestra que:

$$\sum_{\vec{r}} \frac{e^{i(\vec{k}_1-\vec{k}_2)\cdot\vec{r}}}{N} = \delta_{\vec{k}_1,\vec{k}_2} = \delta_{n_{x1},n_{x2}}\,\delta_{n_{y1},n_{y2}}\cdots$$

y

$$\sum_{\vec{k}\in Z.B.} \frac{e^{i\vec{k}\cdot(\vec{r}_1-\vec{r}_2)}}{N} = \delta_{\vec{r}_1,\vec{r}_2} = \delta_{m_{x1},m_{x2}}\,\delta_{m_{y1},m_{y2}}\cdots$$

Transformada de Fourier:

$$\hat{\phi}_{\vec{k}} = \sum_{\vec{r}} e^{i\vec{k}\cdot\vec{r}}\, \phi_{\vec{r}}$$

i) $$\boxed{\phi_{\vec{r}} = \sum_{\vec{k}\in 1.B} \frac{e^{-i\vec{k}\cdot\vec{r}}}{N}\, \hat{\phi}_{\vec{k}}}$$

ii) $$\boxed{\phi_{\vec{r}} \in \mathbb{R} \Rightarrow \hat{\phi}^{*}_{\vec{k}} = \hat{\phi}_{-\vec{k}}}$$

iii) $$\sum_{\vec{r}} \phi_{\vec{r}}^{2} = \sum_{\vec{k}} \frac{1}{N} \left|\hat{\phi}_{\vec{k}}\right|^{2}$$

iv) $$\sum_{\vec{r}} \phi_{\vec{r}}\, \phi_{\vec{r}+d\hat{e}_r}$$

$$\parallel$$

$d = n a_0$, $n \in \mathbb{Z}$

$$\frac{1}{2}\sum_{\vec{r}} \left[\phi_{\vec{r}-d\hat{e}_r}\,\phi_{\vec{r}} + \phi_{\vec{r}}\,\phi_{\vec{r}+d\hat{e}_r}\right]$$

$$= \frac{1}{N} \sum_{\vec{k}}^{+} |\hat{\phi}_{\vec{k}}|^2 \cos(k_x d)$$

imaginemos que:

$$\boxed{Z = \int \prod_{\vec{n}} d\phi_{\vec{n}} \, e^{-S}}$$

donde S es una forma cuadrática de los $\phi_{\vec{n}}$ $\in \mathbb{R}$

$N \times N$

$$S = \sum_{\vec{n},\vec{n}'}^{+} \phi_{\vec{n}} A_{\vec{n}-\vec{n}'} \phi_{\vec{n}'}$$

N $\qquad$ N

inv. por traslación

$$\text{def } \hat{A}_{\vec{k}} = \hat{A}^{*}_{-\vec{k}}$$

$$A_{\vec{n}} = \sum_{\vec{k}}^{+} \frac{1}{N} e^{i\vec{k}\cdot\vec{n}} \hat{A}_{\vec{k}}$$

$$S = \frac{1}{2}(S+S) = \frac{1}{2N}\sum_{\vec{k}} |\hat{\phi}_{\vec{k}}|^2 \left[\hat{A}_{\vec{k}} + \hat{A}^*_{\vec{k}}\right]$$

$$Z = \int \prod_{\vec{n}} d\phi_{\vec{n}}\, e^{-S}$$

$$\hookrightarrow \hat{\phi}_{\vec{n}}$$

Obs $\hat{\phi}_{\vec{n}}$ y $\hat{\phi}_{-\vec{n}}$ no son indep. $\frac{1}{2}$ 2.8.

$$\prod_{\vec{n}} d\phi_{\vec{n}} \to \prod_{\substack{\vec{n} \\ \text{tq } n_x > 0}} d\hat{\phi}^R_{\vec{n}}\, d\hat{\phi}^I_{\vec{n}} = \hat{\prod_{\vec{k}}} d\hat{\phi}_{\vec{k}}$$

N variables

$$\hat{\phi}^R_{\vec{n}} = \mathrm{Re}\left(\hat{\phi}_{\vec{n}}\right)$$

el Jacobiano vale 1.

$$\Rightarrow Z = \int \hat{\prod_{\vec{k}}} d\hat{\phi}_{\vec{n}}\, e^{-\frac{1}{N}\hat{\sum_{k}} |\hat{\phi}_{\vec{n}}|^2 (\hat{A}_{\vec{k}} + \hat{A}^*_{\vec{n}})}$$

$$d\hat{\phi}^I_{\vec{n}}\, d\hat{\phi}^R_{\vec{n}} \qquad \hat{\phi}^{R\,2}_{\vec{n}} + \hat{\phi}^{I\,2}_{\vec{n}}$$

$\rightarrow N$ integrales desacoplados

del estilo $\int_{-\infty}^{\infty} dx\, e^{-\alpha x^2} = \sqrt{\frac{\pi}{\alpha}}$

$$\boxed{Z = \prod_{\vec{k}}^{\Lambda} \frac{\pi N}{2 \operatorname{Re}\{\hat{A}_{\vec{k}}\}}}$$

def: función de correlación:

$$\langle \phi_{\vec{r}_1} \phi_{\vec{r}_2} \rangle$$

$$= \frac{1}{Z} \int \prod_{\vec{k}}^{\Lambda} d\hat{\phi}^{R}_{\vec{k}}\, d\hat{\phi}^{I}_{\vec{k}}\ \phi_{\vec{r}_1} \phi_{\vec{r}_2}\, e^{-S}$$

$$= \frac{1}{Z} \frac{1}{N^2} \sum_{\vec{k}_1, \vec{k}_2} e^{i\vec{k}_1 \cdot \vec{r}_1 + i\vec{k}_2 \cdot \vec{r}_2}$$

$$\times \int \prod_{\vec{k}} d\hat{\phi}_{\vec{k}} \; \hat{\phi}_{\vec{k}_1} \hat{\phi}_{\vec{k}_2} \, e^{-S}$$

$$\hat{\phi}_{\vec{k}_1} = \left(\hat{\phi}^R_{\vec{k}_1} + i \, \hat{\phi}^I_{\vec{k}_1} \right)$$

aqui : $\displaystyle \int dx\,dy \; e^{-\alpha(x^2+y^2)} = \frac{\pi}{\alpha}$

y $\displaystyle \int dx\,dy \; xy\, e^{-\alpha(x^2+y^2)} = 0$

$$\int dx\, x\, e^{-\alpha x^2} = 0$$

~~solo~~ $\displaystyle \int dx\,dy \, (x^2) \, e^{-\alpha(x^2+y^2)} \neq 0$

$$\boxed{\frac{1}{\int dx\,dy\, e^{-\alpha(x^2+y^2)}} \times \int dx\,dy \, (x^2+y^2)\, e^{-\alpha(x^2+y^2)} = \frac{1}{\alpha}}$$

Se muestra que, el único caso que da $\neq 0$ es si $\vec{k}_2 = -\vec{k}_1$!

$$\Rightarrow \boxed{\langle \phi_{\vec{r}_1} \phi_{\vec{r}_2} \rangle = \frac{1}{N^2} \sum_{\vec{k}} e^{i\vec{k}\cdot(\vec{r}_1 - \vec{r}_2)} \langle \hat{\phi}_{\vec{k}} \hat{\phi}_{-\vec{k}} \rangle}$$

$$y \ \langle \hat{\phi}_{\vec{k}} \hat{\phi}_{-\vec{k}} \rangle = \frac{N}{2\mathrm{Re}(A_{\vec{k}})}$$

$$\Rightarrow \boxed{\langle \phi_{\vec{r}_1} \phi_{\vec{r}_2} \rangle = \frac{1}{N} \sum_{\vec{k}} \frac{1}{2\mathrm{Re}\{A_{\vec{k}}\}} e^{i\vec{k}\cdot(\vec{r}_1 - \vec{r}_2)}}$$

Si A acopla los ϕ's en el mismo sitio + primeros vecinos

$$S = \sum_{\vec{r}} \left[\alpha \phi_{\vec{r}}^2 - \beta \left(\phi_{\vec{r}} \phi_{\vec{r}+a_0\hat{e}_x} + \phi_{\vec{r}} \phi_{\vec{r}+a_0\hat{e}_y} + \dots \right) \right]$$

$$S = \frac{1}{N}\sum_{\vec{k}}^{\prime} |\tilde{\phi}_{\vec{k}}|^2 \left[\alpha - \beta\left(\cos(k_x a_0) + \cos(k_y a_0) + \ldots\right)\right]$$

$$= \frac{2}{N}\sum_{k}^{\wedge} \cdot // - \quad - \quad " \quad \text{———} \quad "$$

$$\Rightarrow \boxed{Z = \prod_{\vec{k}}^{\wedge} \frac{\pi N}{2\left[\alpha - \beta\left[\cos(k_x a_0) + \ldots\right]\right]}}$$

$$\boxed{\langle \phi_{\vec{r}_1} \phi_{\vec{r}_2} \rangle = \frac{1}{N}\sum_{\vec{k}}^{\prime} \frac{e^{i\vec{k}\cdot(\vec{r}_1 - \vec{r}_2)}}{2\left[\alpha - \beta\left(\cos(k_x a_0) + \ldots\right)\right]}}$$

2) Límite del continuo

$a_0 \to 0$, L_x, L_y, $\vdots$ finitos, $N_x = \frac{L_x}{a_0} \to \infty$

$N_y \to \infty$

$\vec{k}$ siguiendo

$$\vec{k} = \frac{2\pi n_x}{L_x}\hat{e}_x + \frac{2\pi n_y}{L_y}\hat{e}_y + \ldots$$

Pero $-\infty < n_x < \infty$ etc...

$-\infty < n_y < \infty$

$$\boxed{N = \frac{V}{a_0^d}} \to \infty \,, \quad V = L_x L_y L_z \text{ finito.}$$

diccionario para pasar del espacio discreto al continuo

$$\boxed{\delta_{\vec{r}_1,\vec{r}_2} \longrightarrow a_0^d\, \delta(\vec{r}_1 - \vec{r}_2)}$$

$$\boxed{\sum_{\vec{k}} \longrightarrow \frac{1}{a_0^d}\int d\vec{k}}$$

por ejemplo

$$\sum_{\vec{k}} \frac{e^{i\vec{k}_a(\vec{r}_1-\vec{r}_2)}}{N = \frac{V}{a_0^d}} = \delta_{\vec{r}_1,\vec{r}_2}$$

$$\to \frac{1}{V}\sum_{\vec{k}} e^{i\vec{k}\cdot(\vec{r}_1-\vec{r}_2)} = \delta(\vec{r}_1-\vec{r}_2)$$

y también:

$$\sum_{\vec{r}} \frac{1}{N} e^{i(\vec{k}_1-\vec{k}_2)\cdot\vec{r}} = \delta_{\vec{k}_1,\vec{k}_2}$$

$$\int d\vec{r}\, e^{i(\vec{k}_1-\vec{k}_2)\cdot\vec{r}} = V\,\delta_{\vec{k}_1,\vec{k}_2}$$

$$\phi_{\vec{r}} \Rightarrow \boxed{\phi(\vec{r}) = \frac{1}{a_0^d}\phi_{\vec{r}}}$$

T.F.

$$\hat{\phi}_{\vec{k}} = \sum_{\vec{r}} e^{i\vec{k}\cdot\vec{r}}\phi_{\vec{r}} \Rightarrow \int_V d\vec{r}\, e^{i\vec{k}\cdot\vec{r}}\phi(\vec{r})$$

$$\phi(\vec{r}) = \frac{1}{V}\sum_{\vec{k}} e^{i\vec{k}\cdot\vec{r}}\phi_{\vec{k}}$$

ans:

i) $\sum_{\vec{r}}' \phi_{\vec{r}}^2 \to a_0^d \int d\vec{r}\, \phi^2(\vec{r})$

$$= \frac{a_0^d}{V} \sum_{\vec{k}}' |\hat{\phi}_{\vec{k}}|^2$$

ii) $\sum_{\vec{r}}' \phi_{\vec{r}} \underline{\phi_{\vec{r} \pm a_0 \hat{e}_x}} \to \phi(\vec{r} \pm a_0 \hat{e}_x)$

$$\to a_0^d \int d\vec{r}\, \phi(\vec{r}) \Big(\phi(\vec{r}) \pm a_0 \partial_x \phi(\vec{r}) + \frac{a_0^2}{2} \partial_x^2 \phi(\vec{r}) + \dots\Big)$$

$$\Rightarrow \sum_{r}' \phi_{\vec{r}} \phi_{\vec{r}+a_0\hat{e}_x} + \phi_{\vec{r}} \phi_{\vec{r}-a_0\hat{e}_x}$$

$$\to a_0^d \int d\vec{r} \left[\phi^2(\vec{r}) + a_0^2 \partial_x^2 \phi(\vec{r}) + \dots\right]$$

$$= \frac{a_0^d}{V} \sum_{\vec{k}}' |\hat{\phi}_{\vec{k}}|^2 \left[1 - a_0^2 k_x^2 + \dots\right]$$

$$S = \sum_{\vec{n}}{}' \left[\alpha \phi_n^2 - \beta \left(\phi_{\vec{n}} \phi_{\vec{n}+a_0\vec{e}_\mu} + \dots \right) \right]$$

$$\to \boxed{\int d\vec{r} \left[\frac{a_2}{2} \phi^2(\vec{r}) + \frac{c}{2} (\nabla \phi)^2 + \dots \right]}$$

donde $a_2 = 2a_0^d(\alpha - \beta)$, $c = \beta a_0^{d+2}$

$$S = \frac{1}{V} \sum_{\vec{k}}{}' |\phi_{\vec{k}}|^2 \left(\frac{a_2}{2} + \frac{c\vec{k}^2}{2} \right)$$

obs: $a_2, c > 0$

$$\langle \phi(\vec{r}_1) \phi(\vec{r}_2) \rangle = \frac{1}{a_0^d} \langle \phi_{\vec{n}_1} \phi_{\vec{n}_2} \rangle$$

$$\to \frac{1}{a_0^{2d}} \frac{1}{\left(\frac{V}{a_0^d}\right)} \sum_{\vec{k}}{}' \frac{e^{i\vec{k}\cdot(\vec{r}_1 - \vec{r}_2)}}{2(\alpha - \beta\cos(k_x a_0) + \dots)}$$

pero $\cos(k_x a_0) = 1 - \frac{1}{2} k_x^2 a_0^2 + \dots$

$$\to \boxed{\langle \phi(\vec{r}_1)\phi(\vec{r}_2)\rangle = \frac{1}{V} \times \sum_{\vec{k}}{}' \frac{e^{i\vec{k}\cdot(\vec{r}_1 - \vec{r}_2)}}{a_2 + ck^2}}$$

3] limite de $V \to \infty$

$L_x, L_y, L_z \longrightarrow \infty$

$\to \vec{k}$ variable continue.

$$\Delta k_x \Delta k_y \Delta k_z = \frac{2\pi}{L_x}\frac{2\pi}{L_y}\frac{2\pi}{L_z}$$

$$= \frac{(2\pi)^d}{V}$$

$$\boxed{\sum_{\vec{k}}{}' \to \int \frac{d\vec{k}}{\Delta k_x \Delta k_y \Delta k_z} = V\int \frac{d\vec{k}}{(2\pi)^d}}$$

$$\boxed{\delta_{\vec{k}_1,\vec{k}_2} \longrightarrow \frac{(2\pi)^d}{V}\,\delta(\vec{k}_1-\vec{k}_2)}$$

por ejemplo

$$\sum_{\vec{k}} \frac{e^{i\vec{k}\cdot(\vec{r}_1-\vec{r}_2)}}{V} = \delta(\vec{r}_1-\vec{r}_2)$$

$$\hookrightarrow \int \frac{d\vec{k}}{(2\pi)^d}\, e^{i\vec{k}\cdot(\vec{r}_1-\vec{r}_2)} = \delta(\vec{r}_1-\vec{r}_2)$$

$$\int d\vec{r}\, e^{i(\vec{k}_1-\vec{k}_2)\cdot\vec{r}} = V\,\delta_{\vec{k}_1,\vec{k}_2}$$

$$\hookrightarrow \int d\vec{r}\, e^{i(\vec{k}_1-\vec{k}_2)\cdot\vec{r}} = (2\pi)^d\,\delta(\vec{k}_1-\vec{k}_2)$$

$$\hat{\phi}(\vec{k}) \equiv \int d\vec{r}\, e^{i\vec{k}\cdot\vec{r}}\,\phi(\vec{r})$$

$$\phi(\vec{r}) = \frac{1}{(2\pi)^d}\int d\vec{k}\, e^{-i\vec{k}\cdot\vec{r}}\,\hat{\phi}(\vec{k})$$

$$S = \frac{1}{V} \sum_{\vec{k}} |\hat{\phi}_{\vec{k}}|^2 \left(\frac{a_2}{2} + \frac{c}{2}\vec{k}^2\right)$$

$$\to \int \frac{d\vec{k}}{(2\pi)^d} |\hat{\phi}(\vec{k})|^2 \left(\frac{a_2}{2} + \frac{c\vec{k}^2}{2}\right)$$

$$\Rightarrow \boxed{\langle \phi(\vec{r}_1) \phi(\vec{r}_2) \rangle = \int \frac{d\vec{k}}{(2\pi)^d} \frac{e^{i\vec{k}\cdot(\vec{r}_1 - \vec{r}_2)}}{a_2 + c\vec{k}^2}}$$

$$\langle \phi(\vec{r}_1) \phi(\vec{r}_2) \rangle = G(\vec{r}_1 - \vec{r}_2)$$

dado

$$\boxed{G(\vec{r}) = \tilde{c} \int d\vec{k} \frac{e^{i\vec{k}\cdot\vec{r}}}{\vec{k}^2 + m^2}}$$

$$\tilde{c} = \frac{1}{c(2\pi)^d}, \quad m^2 = \frac{a_2}{c} \geq 0$$

$$G(\bar{r}) \xrightarrow[|\bar{r}| \to \infty]{} \frac{e^{-\frac{r}{\xi}}}{r^{\cdots}}$$

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dónde $\xi = \frac{1}{m}$: la longitud de correlación.