Dynamics of the Universe

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1 Cosmological metric

We have considered the metrics for simple two-dimensional and three-dimensional spaces. Also, we have computed the four-dimensional separation between two events in spacetime through the the *Minkowski metric*. The spatial component of Minkowski spacetime is Euclidean, or flat.

The Minkowski metric of Eq. (40) of GR notes applies only within the context of SR. With no gravity present, Minkowski spacetime is flat and static. When gravity is added, however, the permissible spacetimes are more interesting.

We now consider the rudiments of cosmological theory. The fundamental basis of modern theory is the Friedmann−Robertson−Walker, or hot big bang model.

In the 1920s, the physicist Alexander Friedmann and then in the 1930s, the physicist Howard Robertson and Arthur Walker asked:

Metric of spacetime

"What form can the metric of spacetime assume if the universe is spatially homogeneous and isotropic at all time, and if distances are allowed to expand or contract as a function of time?"

The metric they derived (independently of each other) is called the *Friedmann−Robertson−Walker metric* which is for spherical *comoving* coordinates. It can be written in the form:

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + a(t)^2 \left[\frac{\mathrm{d}r^2}{1-\kappa r^2} + r^2 \mathrm{d}\theta^2 + r^2 \sin(\theta)^2 \mathrm{d}\phi^2 \right], \tag{1}$$

where $a(t)$ is the scale factor, making the metric time dependent. The scale factor, describes how distances in a homogeneous, isotropic universe expand or contract with time.

The time variable t in the Friedmann−Robertson−Walker metric is the cosmological proper time, called the *cosmic time*, and is the time measured by an observer who sees the universe expanding uniformly around him or her.

Calculations 1!

2 The Friedmann equation

The equation that links together $a(t)$, κ, and $\rho(t)$ is known as the *Friedmann equation*, after Alexander Friedmann, the Russian physicist who first derived the equation in 1922. Friedmann actually started his scientific career as a meteorologist. Later, however, he taught himself general relativity, and used Einstein's field equation to described how a spatially homogeneous and isotropic universe expands or contracts as a function of time.

Friedmann published his first results, implying expanding or contracting space, five years before Lemaître interpreted the observed galaxy redshifts in terms of an expanding universe, and seven years before Hubble published Hubble's law.

Friedmann derived his eponymous equation starting from Einstein's field equation, using the full power of general relativity. Even without bringing relativity into play some (thought not all) of the aspects of the Friedmann equation can be understood with the use of purely Newtonian dynamics.

The Friedmann equation is a very important equation in Cosmology. However, if we want to apply the Friedmann equation to the real Universe, we must have some way of tying it to observable properties. For instance, the Friedmann equation can be tied to the Hubble constant, H_0.

Remember, in a universe whose expansion (or contraction) is described by a scale factor $a(t)$, there is a linear relation between recession speed v and proper distance d:

$$v(t) = H(t)d(t),$$

where:

$$H(t) \equiv \frac{\dot{a}}{a}.$$

At the present moment,

$$H_0 = H(t_0) = \left(\frac{\dot{a}}{a}\right)_{t=t_0} = 68 \pm 2\text{kms}^{-1}\text{Mpc}^{-1}.$$

The time− varying function $H(t)$ is generally known as the "Hubble parameter", while H_0, the value of $H(t)$ at the present day, is known as the "Hubble constant".

With the metric given by Eq. (1), we can computing the connection coefficients and curvature tensor to finally get the Einstein's equations.

Calculations 2!

The Friedmann equation become:

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi}{3}\rho - \frac{\kappa}{a^2}. \tag{2}$$

Eq. (2) is known as the Friedman equation, and metrics of the form of Eq. (1) which obeys these equations define FRW universes.

Although the Friedmann equation is indeed important, it can not, all by itself, tell us how the scale factor $a(t)$ evolves with time.

3 The acceleration equation

Even if we had accurate boundary conditions (precise values for ρ_0 and H_0, for instance), it still remains a single equation in two unknowns, $a(t)$ and $\rho(r)$.

We need another equation involving a and ρ if we are to solve for a and ρ as functions of time. The usual form of the acceleration equation is:

The acceleration equation:

$$\frac{\ddot{a}}{a} = -\frac{4\pi}{3}(\rho + 3p). \tag{3}$$

The acceleration equation also includes the pressure p associated with the material filling the universe.

Einstein was interested in finding static ($\ddot{a} = 0$) solutions to account for the astronomical data as they were understood at time.

A static Universe with a positive energy density is compatible with Eq. (19) if the spatial curvature is positive $\kappa = +1$ and the density is appropriately turned. However, Eq. (20) implies that $\ddot{a}$ will never vanish in such spacetime is the pressure p is also nonnegative (which is true for most forms of matter, and certainly for ordinary sources such as stars and gas).

Einstein therefore proposed a modification of his equations, to:

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R + g_{\mu\nu}\Lambda = 8\pi G_{\mu\nu}, \tag{4}$$

where Λ is a new free parameter, the cosmological constant.

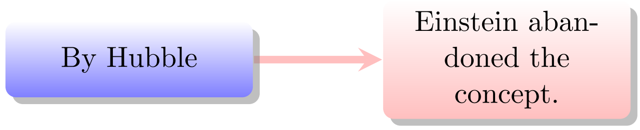
With this modification, the Friedmann equations becomes:

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi}{3}\rho - \frac{k}{a^2} + \frac{\Lambda}{3}. \tag{5}$$

$$\frac{\ddot{a}}{a} = -\frac{4\pi}{3}(\rho + 3p) + \frac{\Lambda}{3}. \tag{6}$$

These equations admit a static solution with positive spatial curvature and all the parameters ρ, p, and Λ nonnegative. This solution is called the *Einstein static Universe*.

Then, the discovery by Hubble that the Universe is expanding eliminated the empirical need for a static world model.



However, the disappearance of the original motivation for introducing the cosmological constant did not change its status as a legitimate addition to the gravitational field equations, or as a parameter to be constrained by observation. Recently, there is reason to believe that Λ is actually nonzero, and Einstein may not have blundered after all.

4 Conservation of energy-momentum tensor

4.1 $\mu = 0$

We are going to consider the components of the conservation of energy-momentum tensor:

$$T^{\mu\nu}_{;\nu} = 0. \tag{7}$$

Or:

$$T^{\mu\nu}_{;\nu} = T^{\mu\nu}_{,\nu} = 0 + \Gamma^{\mu}_{\sigma\nu}T^{\sigma\nu} + \Gamma^{\nu}_{\sigma\nu}T^{\sigma\mu} \tag{8}$$

Calculations 3!

The component $\mu = 0$ becomes:

$$\dot{\rho} + 3\frac{\dot{a}}{a}\rho + 3\frac{\dot{a}}{a}p = 0. \tag{9}$$

Eq. (9) is called the *fluid equation*, and is one of the key equations describing the expansion of the Universe.

For substances of cosmological importance, the equation of state can be written is a simple linear form:

$$p = w\rho, \tag{10}$$

where w is a dimensionless number.

Calculations 4!

The conservation of energy equation Eq.(9) becomes:

$$\frac{\dot{\rho}}{\rho} = -3(1+w)\frac{\dot{a}}{a}. \tag{28}$$

Which can be integrated:

$$\rho \propto a^{-3(1+w)}. \tag{29}$$

The two most popular examples of cosmological fluids are know as radiation and matter.

4.1.1 Radiation

Radiation may be used to described either electromagnetic radiation, or massive particles moving at relative velocities sufficiently close to the speed of light that they become indistinguishable from photons, it obeys:

$$w = {}^1\!/_3.$$

A Universe in which most of the energy density is in the form of radiation is known as radiation−dominated.

The energy density in radiation falls of as:

$$\rho \propto a^{-4}. \tag{11}$$

4.1.2 Dust

Dust is nonrelativistic matter, which obeys:

$$w = 0.$$

Examples include ordinary stars and galaxies, for which the pressure is negligible in comparison with the energy density. Dust is also know as "matter", and universes whose energy density is mostly due to dust are known as matter−dominated.

The energy density in matter falls of as:

$$\rho \propto a^{-3}. \tag{12}$$

This is interpreted as the decrease in the number density of particles as the Universe expands.

Today the energy density of the Universe is dominated by matter, with:

Today:

$$\frac{\rho_{\rm mat}}{\rho_{\rm rad}} \sim 10^6.$$

However, ih the past the universe was much smaller, and the energy density in radiation would have dominated at very early times.

4.1.3 Vacuum

There is one other form of energy that is considered, the vacuum, which:

$$w = -1.$$

The energy density for the vacuum is:

$$\rho \propto \rho_0. \tag{13}$$

So, the energy density is independent of a.

A component of the Universe with $w < -1/3$ is referred to generically as "dark energy". One form of dark energy is of special interest; observational evidence indicates that our universe may contain a *cosmological constant*. A cosmological constant may be defined simply as a component of the universe that has $w = -1$.

4.2 $\mu = i$

Because of isotropy, the spatial components must vanish identically.

$$T^{i\nu}_{;\nu} = T^{i0}_{;0} + T^{i1}_{;1} + T^{i2}_{;2} + T^{i3}_{;3} = 0$$

For example, for $i = 1$.

Calculations 5!

$$T^{1\nu}_{;\nu} = 0.$$

For $i = 2$ and $i = 3$ we have the same result:

$$T^{2\nu}_{;\nu} = 0.$$

$$T^{3\nu}_{;\nu} = 0.$$

5 Solutions to the Friedmann Equations

We are going to see the solutions of the Friedmann equation (19) considering $k = 0$:

$$\frac{da}{a} \propto \sqrt{\rho}\, dt. \tag{14}$$

Calculations 6!

To solve these differential equation we need only to specify the $a(t)$ dependence on the density.

For our epochs:

- Radiation dominated: $\rho \propto a^{-4} \rightarrow a \propto t^{1/2}$.
- Matter dominated: $\rho \propto a^{-3} \rightarrow a \propto t^{2/3}$.
- Vacuum: $\rho \propto \rho_0 \rightarrow a \propto \exp\left(\sqrt{\rho_0}\, t\right)$.

From Eq. (3):

For our epochs:

- The expansion is decelerated:

  Radiation: $a \propto t^{1/2} \rightarrow \ddot{a} < 0$.

  Matter: $a \propto t^{2/3} \rightarrow \ddot{a} < 0$.

- The expansion is accelerated:

  Vacuum: $a \propto \exp\left(\sqrt{\rho_0}\, t\right) \rightarrow \ddot{a} > 0$

If we trace the evolution backwards in time, we necessarily reach a singularity at $a = 0$. This singularity $a = 0$ is the Big Bang. It represents the creation of the Universe from a singular state.

Substituting the various density relations into Eq. (2), we can compare their evolucion. Most of these can be solved in *Mathematica*® through `DSolve` or `NDSolve`.

Notebook...

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