Chapter 1

Dynamical Systems

Deterministic Dynamical Systems

Laws of Nature: many natural phenomena can be understood as *cause-effect* relations

Relations are expressed in mathematical terms: equations, rules, functions.

$x(t)$: state variable of system at time t.

(position, velocity, temperature, pressure, density, wealth, etc.)

before after

$$x(t) \xrightarrow{f} x(t+\tau)$$

f: deterministic rule: set of operations, procedures that allow to calculate the state of a system at time t+τ from the knowledge of its state at time t.

Linear systems: Superposition, $f(x+y) = f(x) + f(y)$, effect proportional to cause.

Behavior is generally regular, periodic, smooth, predictable.

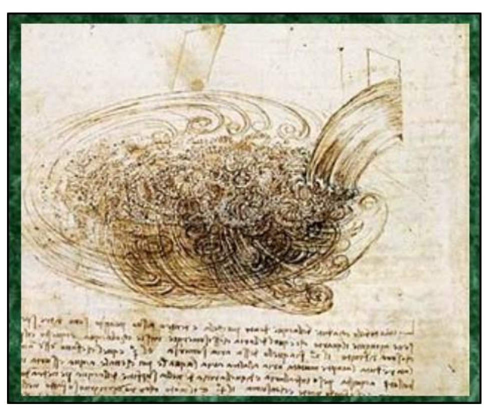
Nonlinear systems: No superposition, $f(x+y) \neq f(x) + f(y)$.

Behavior can be irregular, unpredictable; abundant in Nature.

Classical nonlinear systems: turbulent fluids, unpredictable in time and space. Eqs. are known.

"There is a physical problem common to many fields that is very old and has not being solved…it is the problem of turbulence".

Richard Feynman, *Lectures on Physics,* Vol. I. (1963).



1.1 Brief review of Mechanics

Newton's Laws

Newton's Second Law of motion:

$$\mathbf{F} = m\mathbf{a} \tag{1.1}$$

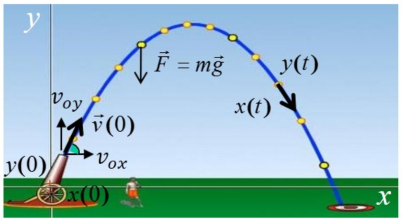
$$\text{Cause} \leftrightarrow \text{Effect}$$

$$\mathbf{F} = m\frac{d\mathbf{v}}{dt} = m\frac{d^2\mathbf{r}}{dt^2}. \tag{1.2}$$

Second order ordinary differential equation for position $\mathbf{r}(t) = (x(t), y(t), z(t))$.
Solution $\mathbf{r}(t)$ is determined by two conditions at $t = 0$: $\mathbf{r}(0)$, $\mathbf{v}(0) = \frac{d(\mathbf{r})}{dt}(0)$.
We use notation: $\dot{x} \equiv \frac{dx}{dt} = v_x$; $\ddot{x} \equiv \frac{d^2x}{dt^2} = \frac{dv_x}{dt}$.

Examples:

1) Proyectile motion.



At $t = 0$: initial position $\mathbf{r}_0 = (x_0, y_0)$, initial velocity $\mathbf{v}_0 = (v_{0x}, v_{0y})$.
Force on particle of mass m in Earth's gravitational field: $\mathbf{F} = m\mathbf{g}$.
In vector components, $\mathbf{F} = (F_x, F_y) = (0, -mg)$.

Newton's Second Law, $\mathbf{F} = m\mathbf{a}$ is a vector equation. For each component:

$$F_x = 0 = m\ddot{x}, \tag{1.3}$$

$$F_y = -mg = m\ddot{y}, \tag{1.4}$$

Solution:

$$x(t) = b_1 t + b_2, \tag{1.5}$$

$$y(t) = -\frac{1}{2}gt^2 + c_1 t + c_2. \qquad b_1,\ b_2,\ c_1,\ c_2 \text{ constants} \tag{1.6}$$

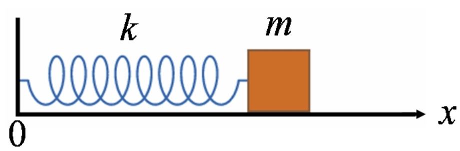
where $b_2 = x_0$, $b_1 = v_{0x}$; $y(0) = c_2 = y_0$ y $\dot{y}(0) = c_1 = v_{0y}$. Then,

$$x(t) = x_0 + v_{0x}t, \tag{1.7}$$

$$y(t) = y_0 + v_{0y}t - \frac{1}{2}gt^2. \tag{1.8}$$

Solution $\mathbf{r}(t) = (x(t), y(t))$ is determined by initial conditions.

2) Harmonic oscillator.



Initial conditions: $x(0)$ and $\dot{x}(0) = v(0)$.
Hooke's Law force: $\mathbf{F} = -k\mathbf{x}$. Then $\mathbf{F} = m\mathbf{a}$ gives:

$$-kx = m\ddot{x} \tag{1.9}$$

can be written as

$$\ddot{x} + \omega^2 x = 0, \qquad \omega^2 \equiv k/m. \tag{1.10}$$

Solution:

$$x(t) = A\sin(\omega t + \phi), \qquad A,\ \phi,\ \text{const.} \tag{1.11}$$

where: $\tan\phi = \omega\dfrac{x(0)}{v(0)}$ and $A^2 = (x(0))^2 + \dfrac{(v(0))^2}{\omega^2}$.

Solution $x(t)$ is determined by initial conditions.

Total mechanical energy = kinetic energy + potential energy:

$$\begin{aligned} E &= \frac{1}{2}mv^2 + \frac{1}{2}kx^2 & (1.12)\\ &= \frac{1}{2}m\dot{x}^2 + \frac{1}{2}kx^2 = \frac{1}{2}mA^2\omega^2 & (1.13)\\ &= \frac{1}{2}m(v(0))^2 + \frac{1}{2}k(x(0))^2 = \text{constant.} & (1.14)\end{aligned}$$

Lagrange's equations

The motion of mechanical systems is described by Newton's Laws. The equations of motion can be formulated in different ways. In the Lagrangian formulation, a mechanical system is represented by a set of s degrees of freedom or generalized coordinates, denoted as $\{q_1, q_2, \ldots, q_s\}$, and a corresponding set of generalized velocities, $\{\dot{q}_1, \dot{q}_2, \ldots, \dot{q}_s\}$. The system can be characterized by a Lagrangian functional $L(q_j, \dot{q}_j, t) = T(q_j, \dot{q}_j, t) - V(q_j, \dot{q}_j, t)$. The equations of motion in the Lagrangian formulation that describe the evolution of the system in time are

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) - \frac{\partial L}{\partial q_j} = 0, \qquad j = 1, \ldots, s. \tag{1.15}$$

Lagrange's equations (1.62) constitute a set of s coupled second-order differential equations for the s generalized coordinates $\{q_j(t)\}$.

Hamilton's equations

The conjugate momentum associate to the coordinate q_j is defined as

$$p_j = \frac{\partial L}{\partial \dot{q}_j} = p_j(q_j, \dot{q}_j, t)\,. \tag{1.16}$$

The Hamiltonian of a system is defined as the function of coordinates, conjugate momenta, and time:

$$H(p_j, q_j, t) = \sum_{j=1}^{s} p_j\dot{q}_j - L(q_j, \dot{q}_j, t)\,, \tag{1.17}$$

where the $\dot{q}_i$ are substituted in terms of the p_i, q_i , and t, by inverting the relations Eq. (1.16), $\dot{q}_j = \dot{q}_j(q_j, p_j, t)$. Then, Hamilton's equations become

$$\dot{q}_i = \frac{\partial H}{\partial p_i}, \quad (1.18)$$

$$\dot{p}_i = -\frac{\partial H}{\partial q_i}, \quad (1.19)$$

Equations (1.18) and (1.19) constitute a system of $2s$ first-order ordinary differential equations respect to time for the q_i y p_i, $i = 1, 2, \ldots, s$, that provide another formulation of the equations of motion for a system.

1.2 Dynamical systems and phase space

The state of many systems (physical, chemical, biological, economic, etc) can be described by a set of n real variables corresponding to n observable quantities (position, pressure, temperature, density, etc) that can change in time. We denote theses variables as $x_i(t)$, $i = 1, \ldots, n$.

The state at a time t can characterized by a vector $\mathbf{x}(t) = (x_1(t), x_2(t), \ldots, x_n(t))$ defined in an Euclidean n-dimensional space $U \in \mathbb{R}^n$ with coordinates $(x_1, x_2, \ldots, x_n)$, called *phase space*, where each coordinate represents a variable of the system.

Deterministic dynamical system: Evolution from $\mathbf{x}(t_1)$ to $\mathbf{x}(t_2)$, $t_2 > t_1$, specified by operational rules (differential equations, iterative functions, algorithms, instructions).

1.3 Dynamical systems with continuous time

In many systems, the evolution of state $\mathbf{x}(t)$ can be described by differential equations:

$$\frac{d\mathbf{x}(t)}{dt} = \mathbf{f}\left(\mathbf{x}(t); \lambda\right), \qquad \lambda\text{: set of parameters.} \quad (1.20)$$

where

$$\mathbf{f}(\mathbf{x}) = \left(f_1(\mathbf{x}), f_2(\mathbf{x}), \ldots, f_n(\mathbf{x})\right), \quad (1.21)$$

Eq. (1.20) corresponds to a system of n first-order ordinary differential equations,

$$\begin{aligned} \dot{x}_1 = &\quad f_1(x_1, x_2, \ldots, x_n), \\ \vdots \quad &\qquad\quad \vdots \\ \dot{x}_n = &\quad f_n(x_1, x_2, \ldots, x_n). \end{aligned} \quad (1.22)$$

In general, a differential equation of order n for one variable can be expressed as a system of n differential equations of first order for n variables.

A solution of system Eq. (1.20) for $\mathbf{x}(t) = (x_1(t), x_2(t), \ldots, x_n(t))$ requires the knowledge of n initial conditions $\mathbf{x}(0) = (x_1(0), x_2(0), \ldots, x_n(0))$.

The system is *linear* if functions $f_i(x_1, x_2, \ldots, x_n)$, $\forall i$ are linear. There exist general methods to find solutions in terms of known functions (polynomial, trigonometric, exponential). Solutions are regular, periodic, smooth, predictable.

A system is *nonlinear* if any function $f_i(x_1, x_2, \ldots, x_n)$ is nonlinear (powers or roots of x_i, products $x_i x_j$), etc). There are no general methods to find solutions $\mathbf{x}(t)$; behavior of $\mathbf{x}(t)$ may greatly change when parameters λ are varied; from regular, periodic, to irregular, unpredictable.

Fixed points or **stationary solutions** $\mathbf{x}^*(\lambda) = (x_1^*, x_2^*, \ldots, x_n^*; \lambda)$ of system Eq. (1.20) are given by

$$\left.\frac{d\mathbf{x}(t)}{dt}\right|_{\mathbf{x}^*} = \mathbf{f}(\mathbf{x}^*; \lambda) = 0. \quad (1.23)$$

Attractor: asymptotic state ($t \to \infty$) of a dynamical system in its phase space.

Examples

1. Hamilton's equations for a mechanical system with s degrees of freedom constitute a $2s$-dimensional dynamical system,

$$\dot{q}_i = \frac{\partial H}{\partial p_i} = f_i(q_j, p_j), \qquad i = 1, \dots, s \tag{1.24}$$

$$\dot{p}_i = -\frac{\partial H}{\partial q_i} = f_{s+i}(q_j, p_j), \qquad i = s+1, \dots, 2s. \tag{1.25}$$

As an example, consider a harmonic oscillator with $s = 1$, whose Lagrangian is

$$L = T - V = \frac{1}{2}m\dot{q}^2 - \frac{1}{2}kq^2. \tag{1.26}$$

The conjugate momentum associated to the coordinate q is

$$p = \frac{\partial L}{\partial \dot{q}} = m\dot{q} \quad \Rightarrow \quad \dot{q} = \frac{p}{m}\,. \tag{1.27}$$

The Hamiltonian is

$$H(q,p) = p\dot{q} - L = p\,\dot{q} - \frac{1}{2}m\,\dot{q}^2 + \frac{1}{2}kq^2 \tag{1.28}$$

$$= \frac{p^2}{2m} + \frac{1}{2}kq^2. \tag{1.29}$$

The corresponding Hamilton's equations are

$$\dot{q} = \frac{\partial H}{\partial p} = \frac{p}{m} = f_1(q,p), \tag{1.30}$$

$$\dot{p} = -\frac{\partial H}{\partial q} = -kq = f_2(q,p). \tag{1.31}$$

The phase space is bidimensional (q,p). The condition

$$f_1(q^*, p^*) = \frac{p^*}{m} = 0, \tag{1.32}$$

$$f_2(q^*, p^*) = -kq^* = 0, \tag{1.33}$$

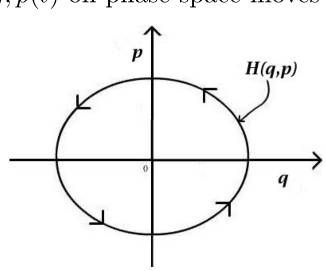
gives the fixed point $(q^*, p^*) = (0,0)$. The system is linear. Solutions :

$$q(t) = A\sin(\omega t + \phi), \qquad p(t) = Am\omega\cos(\omega t + \phi). \tag{1.34}$$

Sustitution of solutions $q(t)$ y $p(t)$ in the Hamiltonian gives

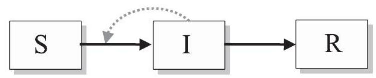
$$H(q,p) = \frac{p^2}{2m} + \frac{1}{2}kq^2 = \frac{m\omega^2 A^2}{2} = \text{cte}. \tag{1.35}$$

The function $H(q,p) = $ const. represents a curve (ellipse) on the two-dimensional phase spáce (q,p). The trajectory described by $q(t), p(t)$ on phase space moves on the elipse $H(q,p) = $ cte.



2. SIR epidemic model.

   $N =$ size of population.
   $S =$ fraction of susceptible individuals in the population.
   $I =$ fraction of infectious individuals in the population.
   $R =$ fraction of recovered individuals in the population.



The dynamics of the basic SIR model is described by the following equations

$$\begin{aligned} \frac{dS}{dt} &= -\beta S I, \\ \frac{dI}{dt} &= \beta S I - \gamma I \\ \frac{dR}{dt} &= \gamma I, \end{aligned} \tag{1.36}$$

where β is the rate of transmission from I to S, γ is the rate of recovery or removal. Then, $\frac{1}{\gamma}$ is the average infectious period. The condition $S + R + I = 1$ means that the total population is conserved (deceased-free model). Initial conditions should be: $S(0) > 0$, $I(0) > 0$, $R(0) = 0$.

The fixed point condition $\dot{S} = 0, \dot{I} = 0, \dot{R} = 0$, corresponds to $I^* = 0, S^* = 0, R^* = 1 - S^* - I^* = 1$.

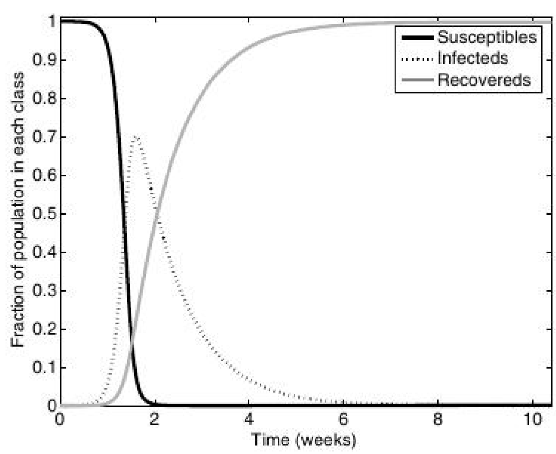


Figure 2.1. The time-evolution of model variables, with an initially entirely susceptible population and a single infectious individual. The figure is plotted assuming $\beta = 520$ per year (or 1.428 per day) and $1/\gamma = 7$ days, giving $R_0 = 10$. (See the following pages for a definition of the crucial

Threshold phenomenon

$$\frac{dI}{dt} = (\beta S - \gamma) I$$

Epidemic dies out when $\frac{dI(0)}{dt} < 0$. That is, when

$$S(0) < \frac{\gamma}{\beta}.$$

The initial fraction of susceptible individuals $S(0)$ should be greater than the critical threshold γ/β (relative removal rate) for the propagation of the epidemic in the population. The inverse quantity $R_o \equiv \frac{\beta}{\gamma}$ is called the *basic reproductive ratio*, defined as *the average number of secondary cases arising from an average primary case in an entirely susceptible population.* This is the most important parameter in epidemic dynamics.

The threshold condition can be written $S(0) < R_o^{-1}$. The threshold condition can also be expressed as follows: if $S(0) = 1$, an epidemic can propagate only if $R_o > 1$.

3. The Lotka-Volterra model describes in a simplified fashion the evolution of two interacting populations, predators and prey, in an ecological system.

Define $n_1(t)$: number of prey (eg. rabbits); $n_2(t)$: number of predators (eg. foxes) at time t. The variation in time $\dot{n}_1$ ($\dot{n}_2$) depends on the gain and loss in the number of prey (predators):

$$\dot{n}_1 = \text{gain}_1 - \text{loss}_1, \tag{1.37}$$
$$\dot{n}_2 = \text{gain}_2 - \text{loss}_2. \tag{1.38}$$

Predator-prey dynamics given by two coupled equations:

$$\dot{n}_1 = \alpha_1 n_1 - \lambda_1 n_1 n_2, \tag{1.39}$$
$$\dot{n}_2 = \lambda_2 n_2 n_1 - \alpha_2 n_2. \tag{1.40}$$

where: α_1 birth rate for prey, λ_1 rate of decreasing prey eaten by predators, α_2 death rate for predators, λ_2 growth rate for predators by eating prey.

If no interaction, $\lambda_1 = \lambda_2 = 0$, prey n_1 would decrease exponentially and predators n_2 would increase exponentially.

Change of variables:

$$m_1 = n_1 \frac{\lambda_2}{\alpha_2}, \qquad m_2 = n_2 \frac{\lambda_1}{\alpha_1}, \tag{1.41}$$

Then, predator-prey Eqs. can be expressed as:

$$\dot{m}_1 = \alpha_1 m_1 (1 - m_2) = f_1(m_1, m_2), \tag{1.42}$$
$$\dot{m}_2 = -\alpha_2 m_2 (1 - m_1) = f_2(m_1, m_2). \tag{1.43}$$

Fixed points : $(m_1^* = 0, m_2^* = 0)$ and $(m_1^* = 1, m_2^* = 1)$.

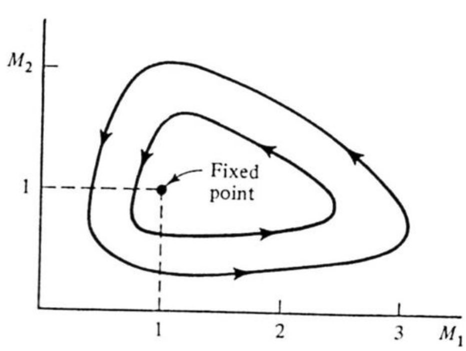
Dividing $\frac{\dot{m}_1}{\dot{m}_2}$, we get:

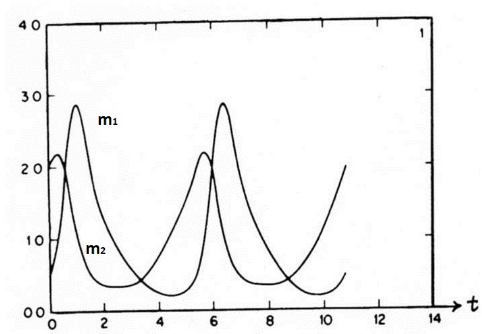
$$\frac{dm_1}{dm_2} = -\left(\frac{\alpha_1}{\alpha_2}\right) \frac{m_1(1 - m_2)}{m_2(1 - m_1)} \tag{1.44}$$
$$\frac{(1 - m_2)}{m_2} dm_2 = -k \frac{(1 - m_1)}{m_1} dm_1 \qquad \text{where } k \equiv \tfrac{\alpha_2}{\alpha_1}. \tag{1.45}$$

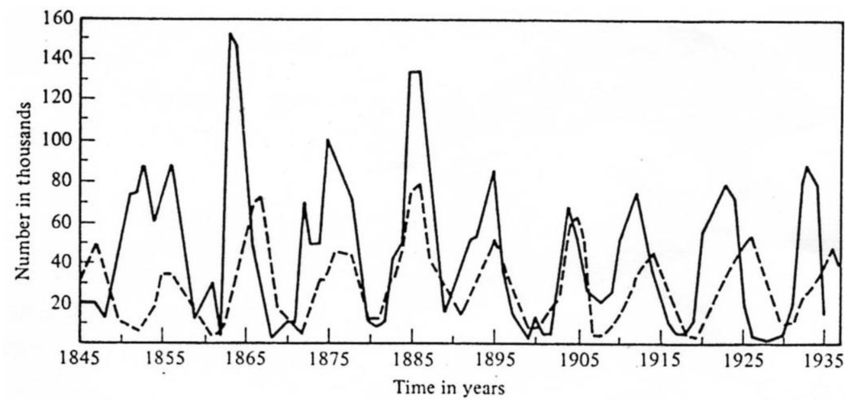
Integrating, we obtain:

$$\begin{aligned} \ln m_2 - m_2 + k \ln m_1 - k m_1 &= \text{const.}, \\ \Rightarrow m_1^k e^{-k m_1} \, m_2 e^{-m_2} &\equiv h(m_1, m_2) = \text{const.} \end{aligned} \tag{1.46}$$

The function $h(m_1, m_2) = $ const represents a closed curve on the two-dimensional phase space (m_1, m_2).







Population of foxes (dashed line) and hares (continuous line) caught by the Hudson Bay Company in Canada in a time lapse of 90 years.

1.4 Existence and Uniqueness Theorem

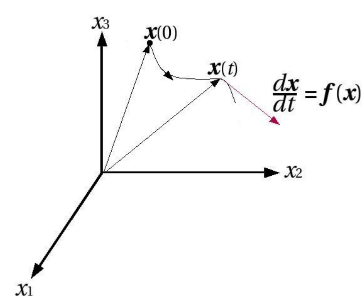
Given a dynamical system described by

$$\frac{d\mathbf{x}(t)}{dt} = \mathbf{f}(\mathbf{x}(t)),$$

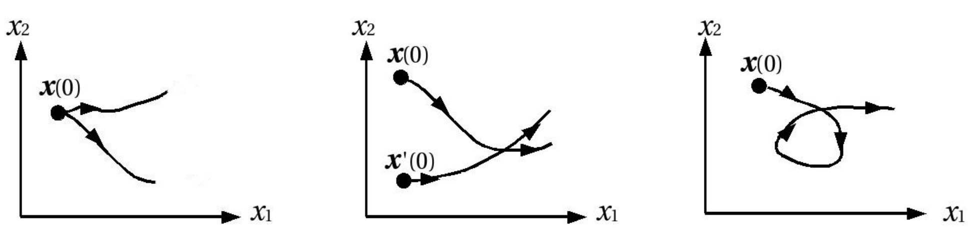
defined on a subspace $U \subseteq \mathrm{Re}^n$, such that $\mathbf{f}(\mathbf{x})$ satisfies the *Lipschitz's property*,

$$|\mathbf{f}(\mathbf{y}) - \mathbf{f}(\mathbf{x})| \leq k\,|\mathbf{y} - \mathbf{x}|\,, \qquad \text{(no singularities)}$$

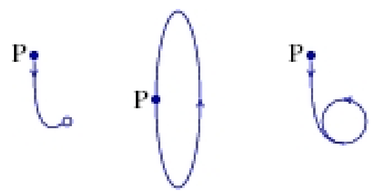
for some $k < \infty$, where $|\mathbf{x}| \equiv \left[x_1^2 + \cdots + x_n^2\right]^{1/2}$, and given a point $\mathbf{x}(0) \in U$, there exists a *unique solution* $\mathbf{x}(t)$ that satisfies this equation for $t \in (0, \tau)$ with initial condition $\mathbf{x}(0)$.



The following situations are *prohibited* by the Uniqueness Theorem in phase space:



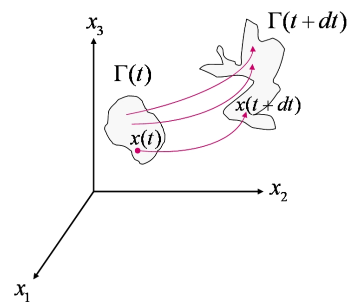
Poincaré-Bendixson Theorem
The only possible asymptotic states in the phase space of a two-dimensional dynamical system are *fixed points* or *limit cycles* (closed trajectories).



The Poincaré-Bendixson Theorem excludes the possibility of irregular or unpredictible behavior in a two-dimensional phase space. It is a consequence of the Existence and Uniqueness Theorem.

1.5 Liouville's Theorem

A set of initial conditions in the phase space of a dynamical system is called an *ensemble*. An ensemble of N initial conditions can be interpreted as a N replicas of a system with different initial conditions, or as a N different realizations of the same system at an initial time.



Each point in the ensemble evolves according to the dynamical equation of the system, Eq. (1.20), given a state $\mathbf{x}(t)$ at a time t. We denote by $\Gamma(t)$ the volumen occupied by the set of points $\mathbf{x}(t)$ on the n-dimensional phase space at time t. Since the system evolves, Γ changes in time. The uniqueness theorem establishes that there cannot arise more than one trajectory from one initial condition. As a consequence, Γ cannot increase in time. That is, the uniqueness theorem implies that a dynamical system should always satisfy

$$\frac{d\Gamma}{dt} \leq 0. \tag{1.47}$$

Suppose that the n-dimensional volume $\Gamma(t)$ is enclosed by an $(n-1)$-dimensional surface S in the phase space of the system. Trajectories in phase space behave analogously to the motion of a fluid of "particles" or a vector field of velocities across the surface S. The change of Γ in time is given by the total number of trajectories that cross (entering or leaving) S per unit time. The flux of a fluid across a surface is defined as the volume of fluid that crosses the surface per unit time. The flux across a differential of area da per unit time is equal to $\mathbf{v} \cdot d\mathbf{a}$, where $\mathbf{v} = \frac{d\mathbf{x}}{dt}$ is the velocity of the incident particle and $d\mathbf{a}$ is the vector normal to the differential of area.

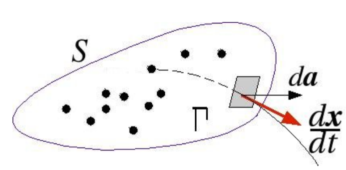


Figure 1.1: Flux of trajectories across a surface S that encloses a volume Γ in phase space.

Then, the change of Γ in time is equal to the total flux per uit time across S,

$$\begin{aligned}\frac{d\Gamma}{dt} &= \oint_S \frac{d\mathbf{x}}{dt}\cdot d\mathbf{a} \qquad \text{(total flux across } S \text{ per unit time)} \\ &= \oint_S \mathbf{f}\cdot d\mathbf{a} = \int_\Gamma \nabla\cdot\mathbf{f}\, d\Gamma' \quad \text{(divergence theorem in } n \text{ dimensions)}, \qquad (1.48)\end{aligned}$$

where

$$d\Gamma' = dx_1 dx_2 \cdots dx_n = \prod_{i=1}^{n} dx_i, \qquad (1.49)$$

is the diferential of volume in the n-dimensional phase space, and

$$\nabla\cdot\mathbf{f} = \nabla\cdot\left(\frac{d\mathbf{x}}{dt}\right) = \sum_{i=1}^{n}\frac{\partial \dot{x}_i}{\partial x_i}. \qquad (1.50)$$

Thus, the condition Eq. (1.47) that every dynamical system satisfies is equivalent to

$$\frac{d\Gamma}{dt} = 0, \quad \text{si} \quad \nabla\cdot\mathbf{f} = 0, \qquad (1.51)$$

$$\frac{d\Gamma}{dt} < 0, \quad \text{si} \quad \nabla\cdot\mathbf{f} < 0. \qquad (1.52)$$

Dynamical systems that satisfy $\nabla\cdot\mathbf{f} = 0$ ($\frac{d\Gamma}{dt} = 0$) are called *conservative*, while systems such that $\nabla\cdot\mathbf{f} < 0$ ($\frac{d\Gamma}{dt} < 0$) are denoted as *dissipative*.

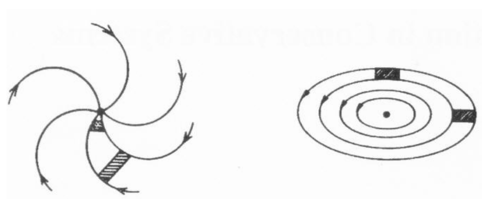


Figure 1.2: Evolution of an ensemble in phase space for a dissipative system (left), and for a conservative system (right).

Trajectories in phase space of dissipative systems converge asymptotically to a geometric object that possesses a smaller volume or lesser dimension than the phase space, it is called an *attractor* of the system. This situation is typical in systems with friction and in out of equilibrium systems.

Liouville's Theorem.
The volume represented by an ensemble of a mechanical system (that obey Hamilton's equations) in phase space is constant, $\Gamma = \text{cte}$, i.e.,

$$\frac{d\Gamma}{dt} = 0.$$

Demostration:
The dynamics of the system is described by Hamilton's equations,

$$\dot{q}_i = \frac{\partial H}{\partial p_i}, \qquad \dot{p}_i = -\frac{\partial H}{\partial q_i},$$

where

$$\mathbf{f} = \left(\frac{\partial H}{\partial p_i}, -\frac{\partial H}{\partial q_i}\right) = \left(\frac{\partial H}{\partial p_1},\ldots,\frac{\partial H}{\partial p_s}, -\frac{\partial H}{\partial q_1},\ldots,-\frac{\partial H}{\partial q_s}\right). \qquad (1.53)$$

Then,

$$\begin{aligned}\nabla\cdot\mathbf{f} &= \sum_i\left(\frac{\partial \dot{q}_i}{\partial q_i} + \frac{\partial \dot{p}_i}{\partial p_i}\right) &(1.54)\\ &= \sum_i\left(\frac{\partial^2 H}{\partial q_i\partial p_i} - \frac{\partial^2 H}{\partial p_i\partial q_i}\right) = 0 &(1.55)\\ &\Rightarrow \frac{d\Gamma}{dt} = 0. &(1.56)\end{aligned}$$

Hamiltonian systems are conservative; the volume of an ensemble may chage its shape but not its size as it moves on phase space. The evolution of an ensemble in the phase space of a Hamiltonian system is similar to the motion of an incompressible fluid in real space.

Examples.

1. Hamilton's equations for the harmonic oscillator are

$$\begin{aligned} \dot{q} &= \frac{p}{m} = f_1(q,p), && (1.57)\\ \dot{p} &= -kq = f_2(q,p), && (1.58) \end{aligned}$$

This system is conservative, since

$$\nabla \cdot \mathbf{f} = \frac{\partial f_1}{\partial q} + \frac{\partial f_2}{\partial p} = 0. \qquad (1.59)$$

2. Lorenz equations, famous in the discovery of the phenomenon of chaos in dynamical systems as we shall see, are

$$\begin{aligned} \dot{x} &= -ax + ay = && f_1(x,y,z),\\ \dot{y} &= -xz + rx - y = && f_2(x,y,z),\\ \dot{z} &= xy - bz = && f_3(x,y,z). \end{aligned} \qquad (1.60)$$

Taking the divergence,

$$\nabla \cdot \mathbf{f} = \frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial z} = -(a+b+1). \qquad (1.61)$$

Thus, Lorenz system is dissipative if $a+b+1>0$. These are the conditions that lead to a chaotic attractor in this system.

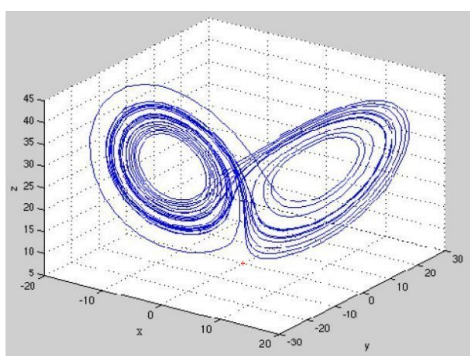


Figure 1.3: Lorenz attractor.

1.6 Integrable and chaotic systems

In the Lagrangian formulation, a mechanical system can be characterized by a function $L(q_j,\dot{q}_j,t) = T(q_j,\dot{q}_j,t) - V(q_j,\dot{q}_j,t)$. The equations of motion that describe the evolution of the system in time are

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) - \frac{\partial L}{\partial q_j} = 0, \qquad j = 1,\ldots,s. \qquad (1.62)$$

Lagrange's equations (1.62) constitute a set of s coupled second-order differential equations for the s generalized coordinates $\{q_j(t)\}$. The conjugate momentum associate to the coordinate q_j is defined as

$$p_j = \frac{\partial L}{\partial \dot{q}_j} = p_j(q_j,\dot{q}_j,t)\,. \qquad (1.63)$$

A coordinate q_i is *cyclic*, i. e., if it does not appear explicitly in the Lagrangian, so that

$$\frac{\partial L}{\partial q_i} = 0. \tag{1.64}$$

Then, we say that the system presents a symmetry related to the motion of the cyclic coordinate; for example, a translation symmetry in space, if q_i corresponds to a spatial dimension; or a rotational symmetry, if q_i describes an angle of rotation about some axis.

If a coordinate q_i is cyclic, the Lagrange's equation for q_i implies that the conjugate momentum $p_i(q_i, \dot{q}_i)$ associated to q_i is a constant. The symmetry is related to a conserved quantity. On the other hand, if the Lagrangian does not explicitly depend on time, the energy of the system is conserved. A system may possess diverse symmetries expressed through its Lagrangian. Noether's theorem establishes that, in general, each symmetry in a system is associated to a conserved quantity.

The conserved quantities represent *first integrals* for the motion of the system; they are functions of the coordinates and velocities (i.e. first-order time derivatives). We denote the set of conserved quantities by $I_k(q_j, \dot{q}_j) = C_k$, where $C_k =$ constante, and $k = 1, \ldots, n$. In general, these quantities can be employed for reducing the number of Lagrange's equations to be integrated.

> A system with s degrees of freedom is *integrable* if it possesses, at least, s conserved quantities; that is, if $n = s$.

The s conserved quantities of an integrable system can be seen as a set of s equations for the s generalized velocities $\dot{q}_j(t)$ which, in principle, can be reduced to a first-order differential equation respect to time for one generalized coordinate. This, in addition, can also be integrated in principle. From the integrated coordinate, the other coordinates may also be integrated. The solutions for all the coordinates $q_j(t)$ can be expressed in terms of *quadratures*; i. e., a set of explicit integrals that generally involve square roots of known functions. However, although the coordinates of integrable systems are susceptible of being completely determined as explicit integrals, the exact calculation of these integrals may not be trivial in many cases.

If a system with s degrees of freedom has less than s conserved quantities ($n < s$), it is called *non-integrable*. A system for which there are more conserved quantities that degrees of freedom ($n > s$), is denoted as *superintegrable*. There are few known superintegrable systems; the simplest example is a free particle; another example is the two-body problem subject to gravitational interaction.

Examples

1. Projectile motion in gravitational field: $s = 2$; $C_1 = E$, $C_2 = p_x$; is integrable.
2. Harmonic oscillator: $s = 1$; $C_1 = E =$ cte, $n = 1$; is integrable.
3. Simple pendulum: $s = 1$; $C_1 = E =$ cte, $n = 1$; is integrable.
4. Double pendulum: $s = 2$; $C_1 = E =$ cte, $n = 1$; is non-integrable.
5. Spring pendulum: $s = 2$; $C_1 = E =$ cte, $n = 1$; is non-integrable.
6. Free particle is superintegrable: $s = 1$; $n = 2$: $C_1 = E =$ cte, $C_2 = p =$ cte.

Integrability may be considered as a type of symmetry present in various dynamical systems, which allows for solutions with regular evolution (periodic or stationary) in time. However, the existence of integrability is not a generic feature; many dynamical system are non-integrable.

A non-integrable system may exhibit a *chaotic* behavior for some ranges of values of their parameters. The phenomenon of *chaos* consists in the extreme sensitivity of the evolution of the variables of a deterministic dynamical system under infinitesimal changes in the initial conditions of those variables. As a consequence, the evolution of the system becomes irregular and unpredictable.

Non-integrability is a necessary but not sufficient condition for the existence of chaos in a system.

Dynamical systems that show chaotic behavior are characterized by the presence of nonlinear functions in the equations describing the evolution of their variables. The double pendulum is an example of a nonlinear and non-integrable system that exhibits chaotic behavior. The Lagrange's equations for the two generalized coordinates of this system, angles θ_1 and θ_2, are

$$\ddot{\theta}_1 = \frac{g(\sin\theta_2\cos\Delta\theta - M\sin\theta_1) - (l_2\dot{\theta}_2^2 + l_1\dot{\theta}_1^2\cos\Delta\theta)\sin\Delta\theta}{l_1(M-\cos^2\Delta\theta)} \tag{1.65}$$

$$\ddot{\theta}_2 = \frac{gM(\sin\theta_1\cos\Delta\theta - \sin\theta_2) - (Ml_1\dot{\theta}_1^2 + l_2\dot{\theta}_2^2\cos\Delta\theta)\sin\Delta\theta}{l_2(M-\cos^2\Delta\theta)}, \tag{1.66}$$

where $\Delta\theta \equiv \theta_1 - \theta_2$, and $M \equiv 1 + m_1/m_2$.

Figure 1.4: Left: Generalized coordinates for the double pendulum. Right: Chaotic motion in space of the particle with mass m_2.

In Chapter 3, we shall explore the chaotic behavior of the double pendulum, and other systems, in detail.