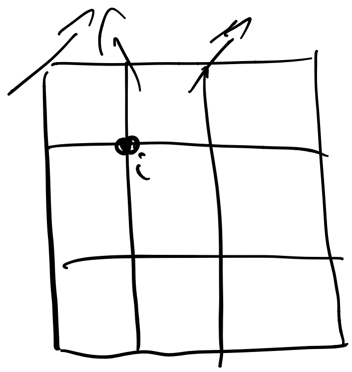
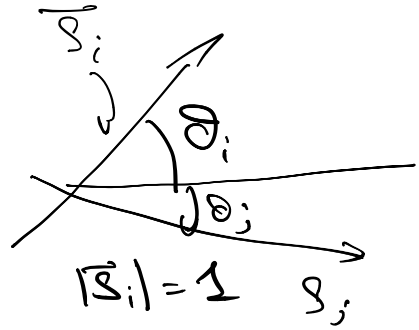
3. Cardy "Scaling and Renormalization ..."

$$\boxed{\text{El modelo XY en 2-D}}$$

a) Acción en el continuo.





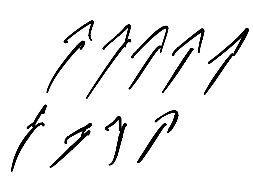
$$\vec{S}_i = S_i^x \hat{e}_x + S_i^y \hat{e}_y = \cos(\theta_i)\hat{e}_x + \sin(\theta_i)\hat{e}_y$$

$$H = -J \sum_{\langle ij \rangle} \vec{S}_i \cdot \vec{S}_j = -J \sum_{\langle ij \rangle}' \cos(\theta_i - \theta_j)$$

$J > 0$

$$Z = \int_0^{2\pi} \prod_i d\theta_i \; e^{\frac{J}{k_B T} \sum_{\langle ij \rangle}' \cos(\theta_i - \theta_j)}$$

A T muy bajo



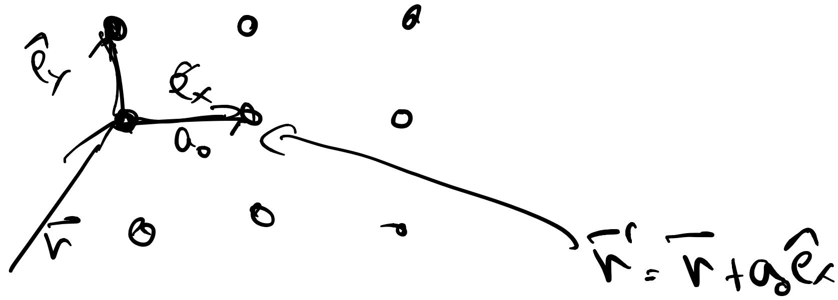
$\theta_i \sim \theta_j \sim \theta_k$

$\Delta\theta_{ij} \sim 0$

$\theta_i \overset{\shortparallel}{-} \theta_j$

$\cos(\Delta\theta_{ij}) \simeq 1 - \frac{1}{2}\Delta\theta_{ij}^2 + \ldots$

$\theta_i , \theta_j \rightarrow \theta(\vec{r})$



$$\theta(\vec{r} + a_0\hat{e}_x) \simeq \underbrace{\theta(\vec{r}) + a_0 \frac{\partial}{\partial x}\theta(\vec{r})}_{} + O(a^2) \ldots$$

$$(\theta_i - \theta_j)^2 \rightarrow \left(\theta(\vec{r}) - \theta(\vec{r} + a\hat{e}_x)\right)^2 \simeq a_0^2\underbrace{\left(\frac{\partial\theta(\vec{r})}{\partial x}\right)^2}_{} + O(a_0^3)$$

$$\sum_{\langle ij \rangle} \rightarrow \frac{1}{a_0^2}\int d^2\vec{r}$$

$$\boxed{-\frac{J}{k_B T}\sum_{\langle ij \rangle} \cos(\theta_i - \theta_j)} \simeq cte + \frac{J}{2k_B T}\sum_{\langle ij \rangle}(\theta_i - \theta_j)^2$$

$$\boxed{\frac{1}{2}\frac{J}{k_B T}\int d^2\vec{r}\,\left(\vec{\nabla}\theta(\vec{r})\right)^2} \text{ etc.}$$

recuerde $S_{GL} = \int d^D\vec{r}\left(\frac{K}{2}(\vec{\nabla}\phi)^2\right)$

$D=2$

b) Funciones de correlación

$$\langle \vec{S}(\vec{0})\cdot\vec{S}(\vec{r})\rangle \sim \langle e^{i\theta(\vec{0})}\,e^{-i\theta(\vec{r})}\rangle$$

$$= \langle e^{i[\theta(\vec{0})-\theta(\vec{r})]}\rangle = e^{-\frac{1}{2}\langle(\theta(\vec{0})-\theta(\vec{r}))^2\rangle}$$

$$= e^{G(\vec{r})-G(\vec{0})}$$

$$G(\vec{r}) = \langle\theta(0)\,\theta(\vec{r})\rangle$$

$$G(\vec{r})-G(\vec{0}) = \frac{Tk_B}{2\pi J}\int d^2\vec{k}\,\frac{e^{i\vec{k}\cdot\vec{r}}-1}{|\vec{k}|^2}$$

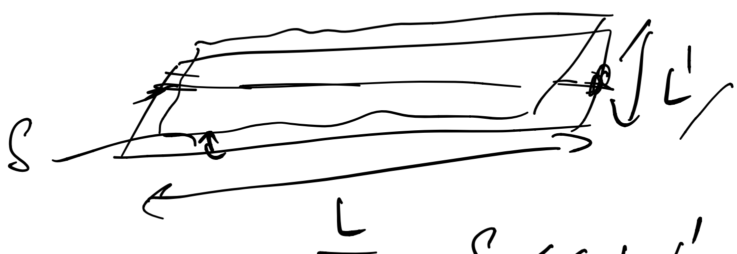
$$= \frac{-T k_B}{2\pi J}\ln\frac{|\vec{r}|}{a_0}$$

$$\Rightarrow \boxed{\left\langle e^{i(\theta(\bar{0})-\theta(\bar{r}))}\right\rangle \sim \text{cte}\left(\frac{a_0}{r}\right)^{\frac{k_B T}{2\pi J}}}$$

→ Quasi-Long-Range-Order

QLRO

c) Superfluidez de capas finas de He^4



$\delta \ll L, L'$

$$\hat{H} = \int d^3\bar{r}\, \hat{\psi}^\dagger(\bar{r})\left(\frac{-\hbar^2\Delta}{2m}\right)\hat{\psi}(\bar{r}) + \frac{1}{2}\int d^3\bar{r}\, d^3\bar{r}'\, \hat{\psi}^\dagger(\bar{r})\,\hat{\psi}^\dagger(\bar{r}')\, V(|\bar{r}-\bar{r}'|)\,\hat{\psi}(\bar{r}')\,\hat{\psi}(\bar{r})$$

→ simetría $U(1)$

$$\forall \bar{r},\quad \hat{\psi}(\bar{r}) \to e^{i\alpha}\hat{\psi}(\bar{r})$$

$$\hat{\psi}^\dagger(\bar{r}) \to e^{-i\alpha}\hat{\psi}^\dagger(\bar{r})$$

→ Condensación de B.E.

$$\langle \hat{\Psi}(\vec{r}) \rangle_{G.S} \neq 0$$

→ ruptura espontanea de sim. $\boxed{a_4 > 0}$

$$S_{GL} = \int d^D\vec{r} \left[\frac{K}{2} |\vec{\nabla}\Psi|^2 + \frac{a_2}{2}|\Psi|^2 + \frac{a_4}{4}|\Psi|^4 + \dots \right.$$

si $a_2 < 0$ $\quad \Psi(\vec{r}) = \rho(\vec{r}) e^{i\theta(\vec{r})} \sim \Psi_0 e^{i\theta(\vec{r})}$

$$\boxed{S_{FL} = cte + \frac{1}{2}\frac{K_0}{T} \int \left(\vec{\nabla}\theta(\vec{r})\right)^2}$$

con $K_0 = K|\Psi_0|^2$

$$\langle \Psi^*(\vec{r})\Psi(\vec{r}) \rangle \sim \frac{cte}{|\vec{r}-\vec{r}'|^{\frac{T}{2\pi K_0}}}$$

la velocidad del condensado superfluido

$$\vec{V}_s(\vec{r}) = \frac{K}{m}\vec{\nabla}\theta(\vec{r})$$

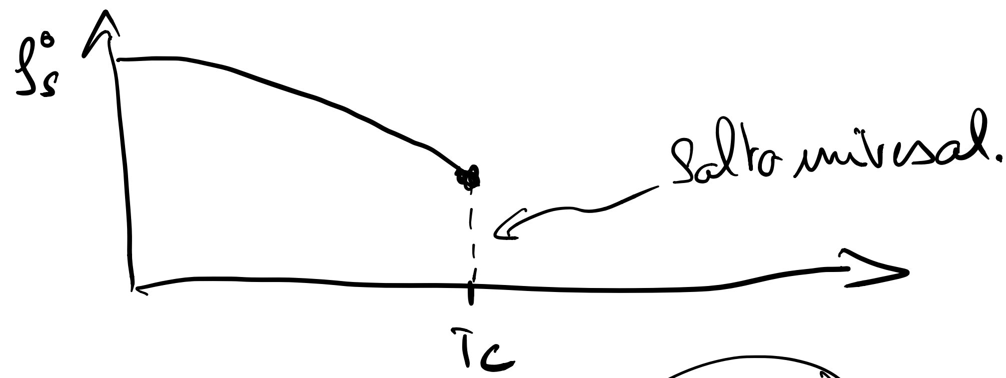
$$\left[= \frac{K}{2mi}\left(\Psi^*\vec{\nabla}\Psi - \Psi\vec{\nabla}\Psi^*\right) \right]$$

$$Z \sim \sum_{\text{conf. fluido}} e^{-\frac{ES}{k_B T}}$$

$$E_s = \frac{1}{2}\rho_s^0 \int d^2\bar{r}\, \bar{V}_s^2(\bar{r})$$

densidad superfluida.

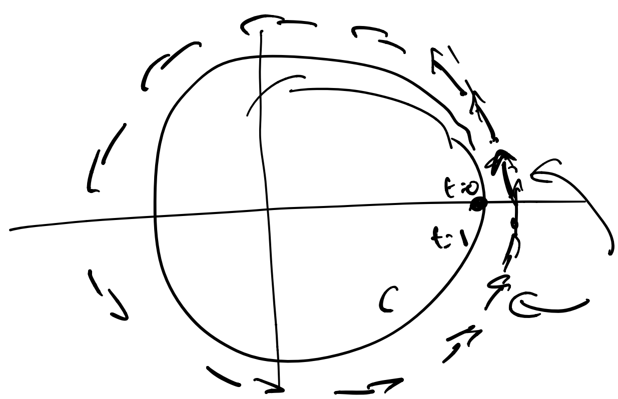
$$\rho_s^0 = \frac{k_B T 4 m^2}{\hbar^2} K_0$$



$$\boxed{\frac{\rho_s^0}{k_B T_c} \simeq \frac{2m^2}{\pi \hbar^2}}$$ (i.e. $K_0 = \frac{2}{\pi}$)

$K_{0\,BKT}$

d) El rol de los vortices



$\Theta(t=1) = \Theta(t=0) + 2\pi$

$$\Rightarrow \oint_C \vec{\nabla}\Theta \cdot \vec{dl} = 2\pi m \quad m \in \mathbb{Z}$$

vortex de vorticidad m

m : invariante topológico.

$\vec{S} = (\cos\theta, \sin\theta, 0)$

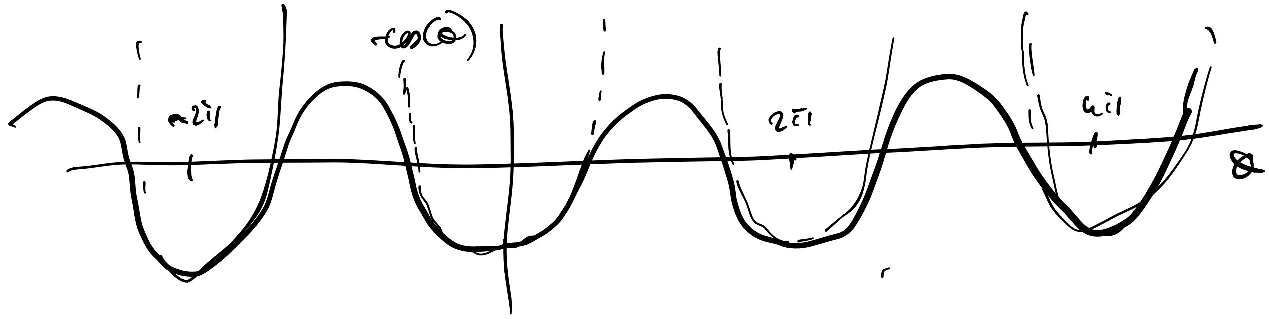
$\partial_\mu \vec{S} = (-\partial_\mu\theta \sin\theta, \partial_\mu\theta \cos\theta, 0)$

$$\vec{S} \wedge \partial_\mu \vec{S} = \partial_\mu\theta \underbrace{(\cos^2\theta + \sin^2\theta)}_{1} \vec{e}_z$$

$$m = \frac{1}{2\pi} \oint_C (\vec{S} \wedge \partial_\mu \vec{S}) \cdot \vec{e}_z \, dl_\mu$$

→ responsables de la transición de fase Berezinskii - Kosterlitz - Thouless.

BKT ó KT.



$$e^{+\cos(\Delta\theta)} \sim \sum_{m=-\infty}^{+\infty} e^{+\frac{1}{2}(\Delta\theta - 2\pi m)^2}$$

→ acción de Vilain

$$\tilde{\beta} = \frac{J}{k_B T} \equiv J\beta$$

$$e^{\tilde{\beta}\cos\theta} = \sum_{n=-\infty}^{+\infty} e^{in\theta}\, I_n(\beta)$$

$$I_n(\beta) = \int_0^{2\pi} e^{\tilde{\beta}\cos\theta}\, e^{-in\theta}\, \frac{d\theta}{2\pi}$$

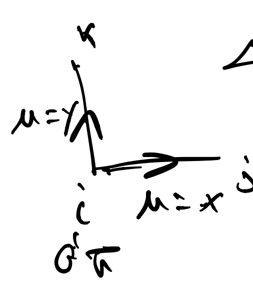
Se muestra que para $\tilde{\beta} \to \infty$ $(T \to 0)$ $\quad e^{\tilde{\beta}} \frac{1}{\sqrt{2\pi\tilde{\beta}}}\, e^{-\frac{n^2}{2\tilde{\beta}}}$

$$e^{\tilde{\beta}\cos\Delta\theta} \xrightarrow[\tilde{\beta}\to\infty]{} \sum_{n=-\infty}^{+\infty} e^{in\Delta\theta}\, \frac{1}{\sqrt{2\pi\tilde{\beta}}}\, e^{-\frac{n^2}{2\tilde{\beta}}}$$

$$Z = \int_0^{2\pi} \prod_i d\theta_i\, e^{\tilde{\beta}\sum_{i,\mu}\cos(\Delta_\mu\theta)}$$

$\Delta_x\theta_i = \theta_j - \theta_i$

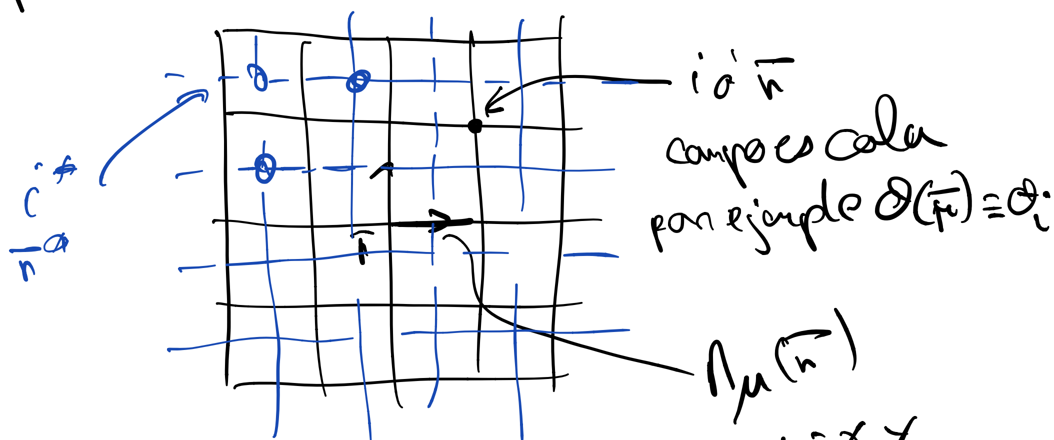
$\Delta_y\theta_i = \theta_k - \theta_i$



$$\Delta_\mu \theta(\vec{n}) = \theta(\vec{n} + a_0 \hat{e}_\mu) - \theta(\vec{n})$$

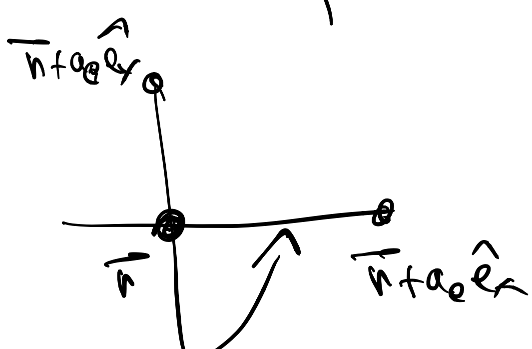
para cada enlace se define un $n_\mu(\vec{n})$

→ campo vectorial en el Lattice.



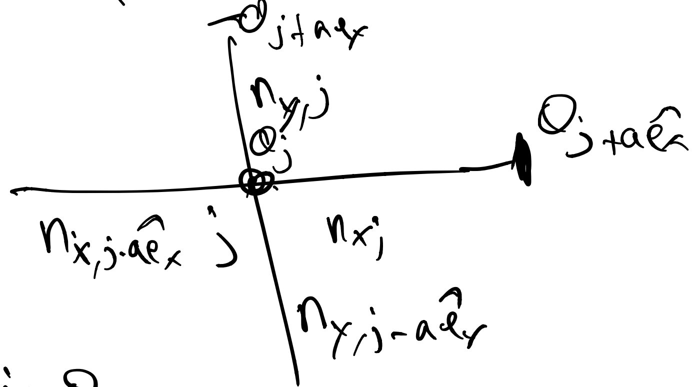
Campo $\phi(\vec{n}^*)$ escalar dual.

$$\theta(\vec{n}) \rightarrow \Delta_\mu \theta(\vec{n}) \qquad \mu = x, y$$



$$\Delta_x \theta(\vec{n}) = \theta(\vec{n} + a_0 \hat{e}_x) - \theta(\vec{n})$$

$$Z = \text{cte} \int_0^{2\pi} \prod_j d\theta_j \sum_{\{n_\mu\}=-\infty}^{+\infty} e^{i n_{j,\mu} \Delta_\mu \theta_j} \, e^{-\frac{n_{j,\mu}^2}{2\tilde{\beta}}}$$



$$\int_0^{2\pi} d\theta \, e^{im\theta} \qquad m \in \mathbb{Z}$$

$= 0$ si $m \neq 0$.

la condicion para que las integrales no se anulen.

$$\forall j \qquad n_{x,j} - n_{x,j-a\hat{e}_x} + n_{y,j} - n_{y,j-a\hat{e}_y} = 0$$

↪ "$\vec{\nabla} \cdot \vec{n}$" $= 0$

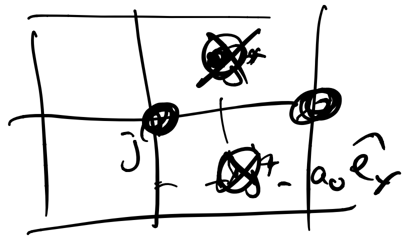
$$\rightarrow \quad n_{\mu,j} = \sum_\nu \epsilon_{\mu\nu} \Delta_\nu P(j^*)$$

$\epsilon_{xy} = 1$ $\quad \epsilon_{xx} = \epsilon_{yy} = 0$

$\epsilon_{yx} = -1$

$P(j^*) \in \mathbb{Z}$

variabilidad.



$P(j^*) - P(j^* - a_0\hat{e}_y) = \Delta_y P(j^*)$

Φ tq $\nabla \cdot \Phi = 0 \rightarrow \partial_\mu \phi_\mu = 0$

$\Rightarrow \phi_\mu = \epsilon_{\mu\nu}\partial_\nu \psi$

$$\Rightarrow Z = \text{cte} \sum_{P(j^*)=-\infty}^{\infty} e^{-\left(\frac{1}{2\tilde{\beta}} \sum_{j^*,\mu} \left(\Delta_\mu P(j^*)\right)^2\right)}$$

Transformación de dualidad.

Fórmula de suma de Poisson.

$$\boxed{\sum_{n=-\infty}^{\infty} g(n) = \sum_{m=-\infty}^{\infty} \int_{-\infty}^{\infty} d\varphi \, g(\varphi) \, e^{2\pi i m \varphi}}$$

$$Z = \mathrm{dre} \int_{-\infty}^{\infty} \prod_{j^*} d\varphi_{j^*} \sum_{m(j^*)=-\infty}^{\infty} e^{-\frac{1}{2\beta}\sum_{j^*,\mu} (\Delta_\mu \varphi_{j^*})^2 + 2\pi i \sum_{j^*} \cdots}$$

$$Z = \int \mathcal{D}\varphi(\vec{r}) \sum_{m(\vec{r})=-\infty}^{\infty} e^{-\frac{1}{2\beta}\int d^2\vec{r}\, (\vec{\nabla}\varphi(\vec{r}))^2 + \frac{2\pi i}{a^2}\int d^2\vec{r}\, m(\vec{r})\varphi(\vec{r})}$$

$$\boxed{Z = Z_0 \sum_{m(\vec{r})=-\infty}^{\infty} e^{\left[-2\pi^2 a^4 \beta \int d\vec{r}\, d\vec{r}\,'\, m(\vec{r})\, G(\vec{r}-\vec{r}\,')\, m(\vec{r}\,')\right]}}$$

$m(\vec{r})$ vórtices en $\vec{r}$ de vorticidad o "carga" $m(\vec{r}) \in \mathbb{Z}$

Z es la función de partición un gas de "partículas" en el ensemble gran-canónico que interaccionan entre sí con el potencial de Coulomb $G(\vec{r}-\vec{r}\,')$

→ gos de Coulomb en 2-D.

obs si $\vec{r}\,' = \vec{r}$

$$\sim e^{-2\pi^2 a^4 \beta\, G(0) \left(\int d^2\vec{r}\, m(\vec{r})\right)^2}$$

$G(0) \to \infty$ para evitar que el termino $\to 0$ se requiere que $\int d\vec{r}\, m(\vec{r}) = 0$

$\to$ vorticidad total $= 0$. !!!

d) analisis por el R.G.

$$Z_V = \sum_{\{m(\vec{r})\}} e^{(-2\pi^2 a^4 \beta \int d^2 r\, d^2 \vec{r}'\, m(\vec{r})\, G(\vec{r}-\vec{r}')\, m(\vec{r}')}$$

$$= e^{-2\pi^2 a^4 \beta \int d^2\vec{r}\, d^2\vec{r}'\, G(\vec{r}-\vec{r}')}$$

$$= e^{-\frac{2\pi\beta}{a^4} \int d\vec{r}\, d\vec{r}' \ln \left|\frac{\vec{r}-\vec{r}'}{a}\right|}$$

$$\sim 1 + \frac{g^2}{a_0^4} \int d^2\vec{r}\, d^2\vec{r}'\, \frac{a_0}{|\vec{r}-\vec{r}'|^{2\pi\beta}}$$

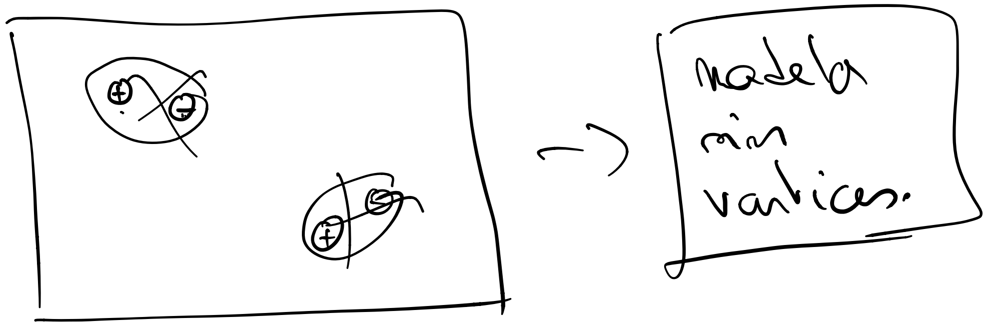
$\vec{r} \to s\vec{r}'$, $a_0 \to s a_0$

$$z_V \to 1 + \frac{\hat{g}^2}{a_0^2}\, s^{4-2\pi\beta} \int d\vec{r}\, d\vec{r}'\, \frac{a_0}{|\vec{r}-\vec{r}'|^{2\pi\beta}}$$

$$\boxed{\hat{g} = s^{4-2\pi\beta}\, g}$$

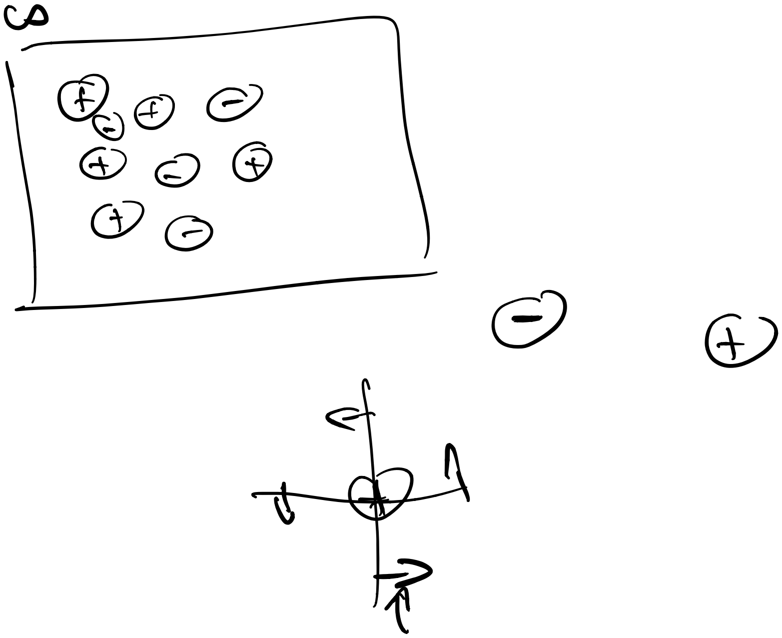
si $4-2\pi\beta<0$ ó $T<T_{BKT}=\frac{\pi J}{2k_B}$

$y\to 0$ los vórtices se hacen cada vez más escasos a grandes escalas.



si $T>T_{BKT}$ ó $4-2\pi\beta>0$

→ los vórtices van a ser cada vez más presentes

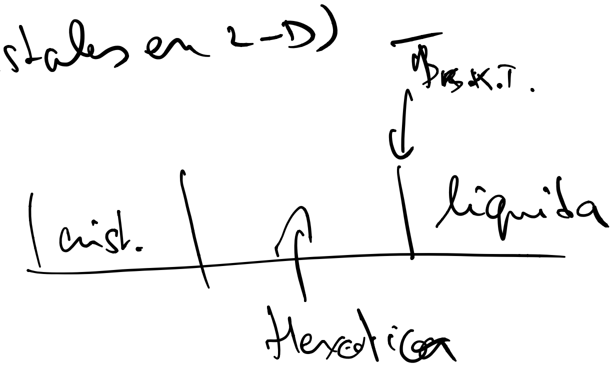


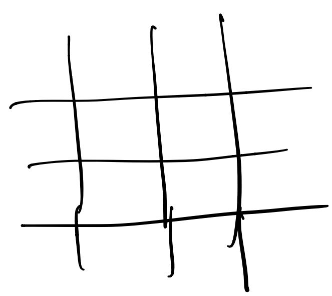
$\hookrightarrow \langle \vec{S}(0)\cdot\vec{S}(\vec{r})\rangle \sim e^{-\frac{|\vec{r}|}{\xi}}$

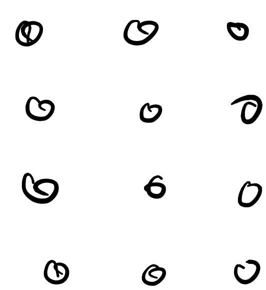
ξ longitud de correlación

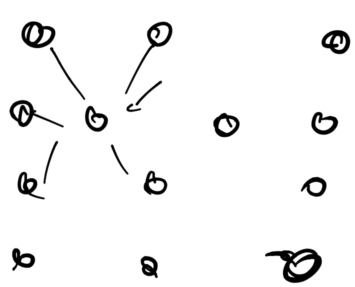
finite $\xi \sim$ espaciamiento típico entre vórtices.

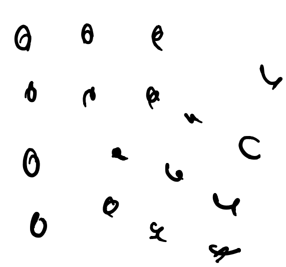
$\Rightarrow$ dislocaciones y disclinaciones (cristales en 2-D)











$$S_{GL} = \frac{J}{k_B T} \sum_{\langle ij \rangle} \cos(\theta_i - \theta_j)$$

$$\sim \frac{J}{2 k_B T} \int d^2 r \, (\nabla \theta)^2$$

$$\boxed{T_{BKT} = \frac{\pi J}{2 k_B}}$$

T

$\sim$ SRO $\langle \vec{S} \cdot \vec{S} \rangle \sim e^{-\frac{|\vec{r} - \vec{r}'|}{\xi}}$

T_{BKT}

QLRO $\langle \vec{S} \cdot \vec{S} \rangle \sim \frac{1}{|\vec{r} - \vec{r}'|^{\frac{k_B T}{2\pi J}}}$

0

Superfluida

3-D